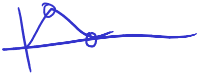
Today

$-u'' = f$



- function spaces
- elliptic PDEs
  - ↳ weak forms
  - ↳ ellipticity
  - ↳ Riesz / Lax-Milgram

Announcements

- Back to Twitch ?
- HW4 out ←

Banach Spaces

$n,m \geq N \quad \|u_n - u_m\| \leq \varepsilon \qquad \|u_n - u\| \to 0$

Definition (Complete/"Banach" space)

Cauchy $\Rightarrow$ Norm conv.

What's special about Cauchy sequences?

Counterexamples?

More on C^0

Let $\Omega \subseteq \mathbb{R}^n$ be open. Is $C^0(\Omega)$ with $\|f\|_\infty := \sup_{x\in\Omega} |f(x)|$ Banach?

$\frac{1}{x}$ ✗ nope.

Is $C^0(\bar{\Omega})$ with $\|f\|_\infty := \sup_{x\in\Omega} |f(x)|$ Banach?

Assume $(f_i)_i$ Cauchy.

- Let $x \in \Omega$. Consider $(f_i(x))_i \to f(x)$. $f \in C^0(\bar{\Omega})$?
- Let $\varepsilon > 0$. There exists an N so that $|f_n(x) - f_m(x)| < \varepsilon$ for $m, n \geq N$. Take the limit $m \to \infty$

  $|f_n(x) - f(x)| < \varepsilon$. conv ✓ By uniform conv, continuity of the limit.

C^m Spaces

Let $\Omega \subseteq \mathbb{R}^n$. closed

Consider a multi-index $\boldsymbol{k} = (k_1, \ldots, k_n)$ $\in \mathbb{N}_0^n$ and define the symbols

$$D^{\vec{k}} f = \frac{\partial^{|\vec{k}|}}{\partial^{k_1}_{x_1} \cdots \partial^{k_n}_{x_n}} f. \qquad |\vec{k}| = k_1 + \cdots + k_n$$

Definition (C^m Spaces)

$$C^m(\Omega) = \{ f \in C^0(\Omega) \; : D^{\vec{k}} f \in C^0(\Omega) \text{ for } |\vec{k}| \leq m \}$$

$$C^\infty(\Omega) = \{ f \in C^0(\Omega) \; : D^{\vec{k}} f \in C^0(\Omega) \text{ for all } k \}$$

$$C^m_0(\Omega) = \{ f \in C^m(\Omega) : f \text{ has compact supp} \}$$

f comp. supp $\Leftrightarrow$ there exists a compact (cl. & bdd) set S so that $f(x) = 0$ if $x \notin S$.

L^p Spaces

Let $1 \le p < \infty$.

Lebesgue integral

Definition (L^p Spaces)

$$L^p(\Omega) := \left\{ u : (u : \mathbb{R} \to \mathbb{R}) \text{ measurable}, \int_\Omega |u|^p \, dx < \infty \right\},$$

$$\|u\|_p := \left(\int_\Omega |u|^p \, dx \right)^{1/p}.$$

except at a set M w/
$0 = \mu(M$

Definition (L^∞ Space)

$$L^\infty(\Omega) := \{ u : (u : \mathbb{R} \to \mathbb{R}), |u(x)| < \infty \text{ almost everywhere} \},$$

$$\|u\|_\infty = \inf \{ C : |u(x)| \le C \text{ almost everywhere} \}.$$

$\mu(M) = \int_M 1$

L^p Spaces: Properties

$(u, v) \le \|u\|_{L^2} \|v\|_{L^2}$

Theorem (Hölder's Inequality)

For $1 \le p, q \le \infty$ *with* $1/p + 1/q = 1$ *and measurable* u *and* v,

$$\boxed{\|uv\|_1 \le \|u\|_p \|v\|_q .}$$

Theorem (Minkowski's Inequality (Triangle inequality in L^p))

For $1 \le p \le \infty$ *and* $u, v \in L^p(\Omega)$,

$$\boxed{\|u+v\|_p \le \|u\|_p + \|v\|_p}$$

Inner Product Spaces

Let V be a vector space.

Definition (Inner Product)

An inner product is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ such that for any $f, g, h \in V$ and $\alpha \in \mathbb{R}$

$$\begin{aligned}
\langle f, f \rangle &\geq 0, \\
\langle f, f \rangle &= 0 \Leftrightarrow f = 0, \\
\langle f, g \rangle &= \cancel{\langle f, g \rangle}, \langle g, f \rangle \\
\langle \alpha f + g, h \rangle &= \alpha \langle f, h \rangle + \langle g, h \rangle .
\end{aligned}$$

$(\cdot, \cdot)$

Definition (Induced Norm)

$$\|f\| = \sqrt{\langle f, f \rangle}.$$

Hilbert Spaces

Definition (Hilbert Space)

An inner product space that is complete under the induced norm.

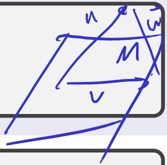
Let Ω be open.

Theorem (L^2)

$L^2(\Omega)$ equals the closure of (set of all imits of Cauchy sequences in) $C_0^\infty(\Omega)$ under the induced norm $\|\cdot\|_2$.

Theorem (Hilbert Projection)

Let $M \subseteq V$ (Hilbert) closed subspace. For any $u \in V$
$\exists$ unique $u = v + w$ $\quad v \in M \quad w \in M^\perp$



$M^\perp = \{ w \in V : \langle z, w \rangle = 0 \text{ for all } z \in M \}$

Weak Derivatives

Define the space L^1_{loc} of locally integrable functions. $\int |u| < \infty$

$$L^1_{loc}(\Omega) = \{ u : (\Omega \to \mathbb{R}) \text{ meas.}$$

$$\int |u(x)\varphi(x)| < \infty, \ \varphi \in C^\infty_0(\Omega) \}$$

$$L^1 \subseteq L^1_{loc}$$

Definition (Weak Derivative)

$v \in L^1_{\text{loc}}(\Omega)$ is the weak partial derivative of $u \in L^1_{\text{loc}}(\Omega)$ of multi-index order $\boldsymbol{k}$ if

$$\int_\Omega \underbrace{v}_{\hookrightarrow D^k u} \varphi \, dx = (-1)^k \int_\Omega u \, D^k \varphi \, dx \quad \text{for all } \varphi \in C^\infty_0(\Omega)$$

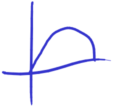
In this case $D^{\boldsymbol{k}} u := v$

Weak Derivatives: Examples (1/2)

Consider all these on the interval $[-1, 1]$.

"$D_w^{\vec{k}}$" ← weak deriv.

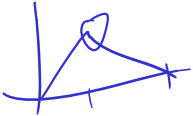
$$f_1(x) = 4(1-x)x$$



strong diff ⇒ weak diff.

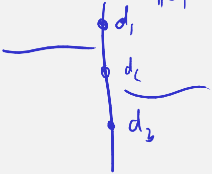
$D_w f_1(x) = 4 - 8x$

$$f_2(x) = \begin{cases} 2x & x \leq 1/2, \\ 2 - 2x & x > 1/2. \end{cases}$$



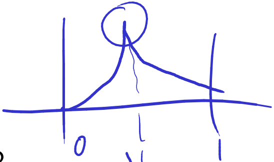
yes, kinks are OK

$d_1 = d_2 \iff \|d_1 - d_2\|_1 = 0$



$D_w f_2(x) = \begin{cases} 2 & x \leq \frac{1}{2} \\ -2 & x > \frac{1}{2} \end{cases}$

Weak Derivatives: Examples (2/2)



$$f_3(x) = \sqrt{\frac{1}{2}} + \begin{cases} -\sqrt{\frac{1}{2} - x} & x \le 1/2, \\ -\sqrt{x - \frac{1}{2}} & x > 1/2. \end{cases}$$

(Some) cusps are OK:

$$D_w f_3(x) = \begin{cases} \dfrac{1}{2\sqrt{1/2 - x}} & x \le \frac{1}{2} \\ -\dfrac{1}{2\sqrt{x - \frac{1}{2}}} & x > \frac{1}{2} \end{cases}$$

Sobolev Spaces

Let $\Omega \subset \mathbb{R}^n$, $k \in \mathbb{N}_0$ and $1 \leq p < \infty$.

Definition ((k, p)-Sobolev Norm/Space)

$$\|u\|_{k,p} = \sqrt[p]{\sum_{|\vec{\alpha}| \leq k} \|D_w^{\vec{\alpha}} u\|_p^p}$$

$$W_{k,p}(\Omega) = \{ u : (u : \Omega \to \mathbb{R}) \text{ with } \|u\|_{k,p} < \infty \}$$

More Sobolev Spaces

$W^{0,2}$?

L^2

$W^{s,2}$? $s \in \mathbb{N}_0$

$H^s = W^{s,2}$. $\leftarrow$ Hilbert spaces

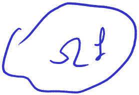
H^1

$H_0^1(\Omega)$?

Closure of $C_0^\infty(\Omega)$ under $\|\cdot\|_{1,2}$.

Outline

An Elliptic Model Problem



Let $\Omega \subset \mathbb{R}^n$ open, bounded, $f \in H^1(\bar{\Omega})$.

$$\begin{aligned} -\nabla \cdot \nabla u + u &= f(x) \quad (x \in \Omega), \\ u(x) &= 0 \quad (x \in \partial\Omega). \end{aligned}$$

$\Delta u = \nabla \cdot \nabla u = \operatorname{div} \operatorname{grad} u = (\partial_{x_1}^2 + \dots + \partial_{x_n}^2) u$

Let $V := H^1(\Omega)$. Integration by parts? (Gauss's theorem applied to $u\boldsymbol{v}$):

$$\int_\Omega \nabla u \cdot \vec{v} + \int_\Omega u \nabla \cdot \vec{v} = \int_\Omega \nabla \cdot (u \vec{v}) = \int_{\partial\Omega} (u\vec{v}) \cdot \hat{n}$$

$v = \nabla u$
$u = \varphi$

Weak form?

$$\int -\nabla \cdot \nabla u \, \varphi \, dx + \int u \varphi \, dx = \int f \varphi \, dx$$

$$\int_\Omega \nabla u \cdot \nabla \varphi + \underbrace{\int_{\partial\Omega} \hat{n} \cdot (\nabla u \, \varphi)}_{\varphi = 0 \Rightarrow 0} + \int_\Omega u \varphi = \int f \varphi \quad \leftarrow \text{weak form}$$

$u \in H^1_0 \rightsquigarrow \varphi \in H^1_0$

$-\Delta u = f$ "Poisson"

$-\Delta u + cu = f$ "Yukawa" $c > 0$

Motivation: Bilinear Forms and Functionals

$$\int_\Omega \nabla u \cdot \nabla v + \int_\Omega uv = \int fv.$$

Recast this in terms of bilinear forms and functionals:

bilinear form $\to a(u,v) = \int_\Omega \nabla u \cdot \nabla v + \int uv$

linear functional $\to g(v) = \int fv = (f,v)_{L^2}$.

$a(u,v) = g(v)$ For all $v \in H_0^1(\Omega)$

Dual Spaces and Functionals

$g(x+y) = g(x) + g(y)$

Bounded Linear Functional

Let $(V, \|\cdot\|)$ be a Banach space. A linear functional is a linear function $g : V \to \mathbb{R}$. It is bounded ($\Leftrightarrow$ continuous) if there exists a constant C so that $|g(v)| \leq C \|v\|$ for all $v \in V$.

Dual Space

Let $(V, \|\cdot\|)$ be a Banach space. Then the dual space V' is the space of bounded linear functionals on V.

Dual Space is Banach (cf. e.g. Trèves 1967)

V' is a Banach space with the dual norm

$$\|g\|_{V'} = \sup_{v \in V \setminus \{0\}} \frac{|g(v)|}{\|v\|_V}.$$

Functionals in the Model Problem

$g(v) = \langle f, v \rangle$

Is g from the model problem a bounded functional? (In what space?)

Must use $V = H^1$ (because that's where the problem lives).

$$\|g\|_{V'} = \sup \frac{|\langle f, v\rangle_{L^2}|}{\|v\|_{H^1}} \le \frac{\|f\|_{L^2}\|v\|_{L^2}}{\|v\|_{L^2} + \underbrace{\|D_w f\|_{L^2}}} \le \frac{\|f\|_{L^2}\cancel{\|v\|_{L^2}}}{\cancel{\|v\|_{L^2}}} \le \|f\|_{L^2}$$

That bound felt loose and wasteful. Can we do better?