Today

- 207E assembly

Announcements

- HW5 out
- Project 1 due

A Boundary Value Problem

Consider the following elliptic PDE

$$-\nabla \cdot (\kappa(\boldsymbol{x})\nabla u) = f(\boldsymbol{x}) \quad \text{for } \boldsymbol{x} \in \Omega \subset \mathbb{R}^2,$$
$$u(\boldsymbol{x}) = 0 \quad \text{when} \quad \boldsymbol{x} \in \partial\Omega.$$

Weak form?

$u \in H_0^1(\Omega)$, $v \in H_0^1(\Omega)$:

$$\int_\Omega -\nabla\cdot(\kappa \nabla u)\, v = \int_\Omega f v$$

$$\cancel{-\int v\, \hat{n}\cdot(\kappa\nabla u)} + \int_\Omega \kappa(x)\nabla u\cdot\nabla v = \int_\Omega f v$$

$v \in H_0^1$

Weak Form: Bilinear Form and RHS Functional

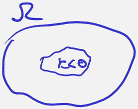
Hence the problem is to find $u \in V$, such that

$$a(u, v) = g(v), \quad \text{for all } v \in V = H_0^1(\Omega)$$

where...

$$a(u,v) = \int_\Omega \kappa(x) \cdot (\nabla u(x) \cdot \nabla v(x))\, dx$$

$$g(v) = \int_\Omega f(x)\, v(x)\, dx$$



Is this symmetric, coercive, and continuous?

symm: ✓

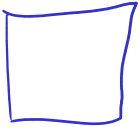
coercive: $0 < c \leq \kappa$ $\quad \hookrightarrow a(u,u) \geq c_0 \|u\|_{H^1}$

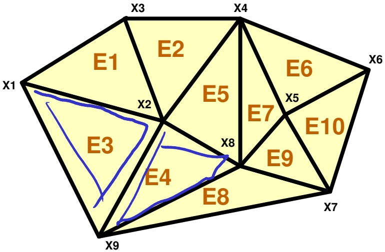
continuity: $|\kappa| \leq C$ $\quad \hookrightarrow \|u\|_{L^2} \leq \|\nabla u\|_{L^2}$ P-F ineq.

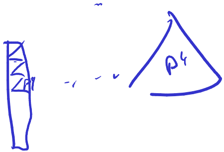
$u \in H_0^1$

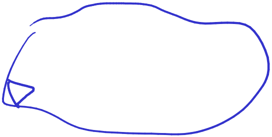
Triangulation: 2D

Suppose the domain is a union of triangles E_m, with vertices x_i.









$$\Omega = E_1 \cup E_2 \cup \cdots \cup E_{10}$$

Elements and the Bilinear Form

If the domain, Ω, can be written as a disjoint union of elements, E_k,

$$\Omega = \cup_{m=1}^{M} E_m \quad \text{with} \quad E_i^\circ \cap E_j^\circ = \emptyset \text{ for } i \neq j,$$

what happens to a and g?

$$a(u,v) = \sum_{m=1}^{M} \int_{E_m} K(x) \, (\nabla u \cdot \nabla v)$$

$$g(v) = \sum_{m=1}^{M} \int_{E_m} f(x) \cdot v(x) \, dx .$$

Basis Functions

Expand

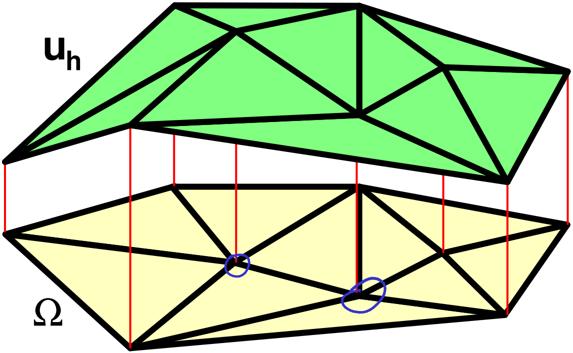
$$u_N(\boldsymbol{x}) = \sum_{i=1}^{N_p} u_i \varphi_i,$$

and plug into the weak form.

$$\sum_{j=1}^{N_p} u_j \, a(\varphi_j, \varphi_i) = g(\varphi_i)$$

Global Lagrange Basis

Approximate solution u_h: Piecewise linear on Ω



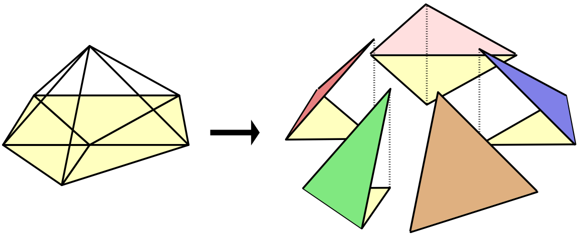
The Lagrange basis for S_h consists of piecewise linear φ_i, with...

$\varphi_i(x_i) = 1 \qquad \varphi_i(x_j) = 0 \quad (j \neq i)$

Basis Functions Features

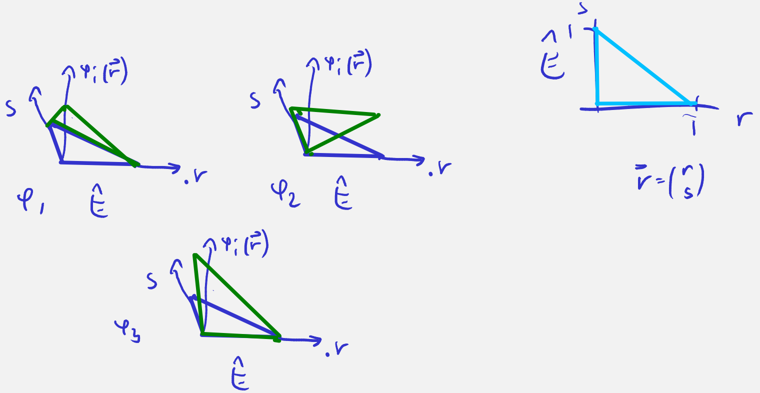
Features of the basis?

- $\varphi_i \in C^0$, $\varphi_i \in H^1$
- Restricted to each el, φ_i are linear.



Local Basis

What basis functions exist on each triangle?



Local Basis Expressions

Write expressions for the nodal linear basis in 2D.

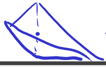
$\varphi_1(r,s) = 1 - r - s$

$\varphi_2(r,s) = r$

$\varphi_3(r,s) = s$

Higher-Order, Higher-Dimensional Simplex Bases

⊢ = 1-simplex



= tet = 3-simp.

$\vec{r} = \begin{pmatrix} r \\ s \end{pmatrix}$ What's an n-simplex? Δ = 2- simplex

$\hookrightarrow$ $r_i \geq 0$ $\quad \sum r_i \leq 1$ $\quad$ ($\to$ barycentric)

Give a higher-order polynomial space on the n-simplex:

High order on the Δ:

$$\begin{array}{ccc} 1 & & \\ r & s & \\ r^2 & rs & s^2 \end{array}$$

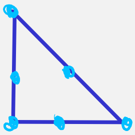
$$P^N(\Delta) = \text{span}\{ r^i s^j : 0 \leq i+j \leq N \}$$

P^1 ⤴

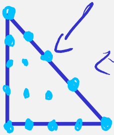
$$P^N(n\text{-simplex}) = \text{span} \left\{ \prod_{i=1}^{n} r_i^{d_i} : \sum d_i \leq N \right\}$$

Give nodal sets (on the $\triangle$) for P^N and dim P^N in general.

$P2$

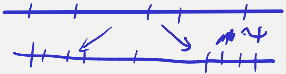


$P4$



? 15 ✓

$$\dim P^N = \frac{(N+1)(N+2)}{2}$$



Finding a Nodal/Lagrange Basis in General

Given a nodal set $(\xi_i)_{i=1}^{N_p} \subset \hat{E}$ (where $\hat{E}$ is the reference element) and a basis $(\varphi_j)_{j=1}^{N_p} : \hat{E} \to \mathbb{R}$, find a Lagrange basis.

$$V = \begin{bmatrix} \varphi_1(\xi_1) & & \varphi_{N_p}(\xi_1) \\ \vdots & & \\ \varphi_1(\xi_{N_p}) & \cdots & \varphi_{N_p}(\xi_{N_p}) \end{bmatrix}$$

$$\ell_i = \sum_{j=1} \left(V^{-T}\right)_{i,j} \varphi_j$$

In the tensor product case?

Higher-Order, Higher-Dimensional Tensor Product Bases

What's a tensor product element? $[0,1]^n$

$\leftarrow$ $0 \le r_i \le 1$

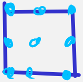
Give a higher-order polynomial space on the *n*-~~simplex~~: TP el.

$$Q^N = \left\{ \prod r_i^{d_i} : 0 \le d_i \le N \right\}$$

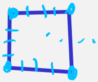
↳ Lagrange basis can be found dim-by-dim

Give the nodal sets (on the quad) for Q^N.

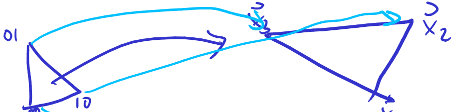
Q^2:



Q^4:



Element Mappings



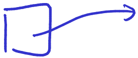
Construct a mapping $T_m : \hat{E} \to E_m$ that takes the triangular reference element $\hat{E}$ to a global triangle E_m.

$$T_m(r,s) = (\vec{x}_2 - \vec{x}_1)\, r + (\vec{x}_3 - \vec{x}_1)\, s + \vec{x}_1 = \underbrace{\left(\vec{x}_2 - \vec{x}_1,\ \vec{x}_3 - \vec{x}_1\right)}_{J_{T_m}} \vec{r} + x_1$$

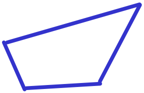
What is the Jacobian of T_m?

$$J_{T_m}(r,s) = \left(\vec{x}_2 - \vec{x}_1,\ \vec{x}_3 - \vec{x}_1\right)$$

More on Mappings



$T_m(\vec{r}) = A\vec{x} + B$



Is an affine mapping sufficient for a tensor product element?

Affine maps take □ to parallelograms

Affine: components $x, y \in P^1$

TP: components $x, y \in Q^1$

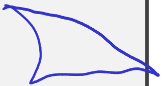
$T_m : \underset{\subset \mathbb{R}^2}{\hat{E}} \to \mathbb{R}^2 \quad \begin{pmatrix} x(r,s) \\ y(r,s) \end{pmatrix}$

How might we accomplish curvilinear elements using the same idea?

Choose $T_m \in (P^N)^n$ "isoparametric"

"sub"

"super"



Constructing the Global Basis

Construct a basis on the element E_m from the reference basis $(\hat{\varphi}_j)_{j=1}^{N_p} : E_m \to \mathbb{R}$.

$$\hat{\varphi}_{m,j}(\vec{x}) = \hat{\varphi}_j(T_m^{-1}(\vec{x}))$$

What's the gradient of this basis?

$$\nabla_x \hat{\varphi}_{m,j} = \left(\frac{d}{dx}\hat{\varphi}_j(T_m^{-1}(\vec{x}))\right)^T$$

$$= \left(\left.\frac{d\hat{\varphi}_j}{dr}\right|_{\vec{r}=T^{-1}(\vec{x})} \cdot J_T^{-1}(\vec{x})\right)^T$$

$$= J_T^{-T}(\vec{x}) \cdot \nabla_{\vec{r}}\hat{\varphi}_j(T^{-1}(\vec{x}))$$

Assembling a Linear System

Express the matrix and vector elements in

$$\sum_{j=1}^{N_p} u_j a(\varphi_j, \varphi_i) = g(\varphi_i) \quad \text{for } i = 1, \ldots, N_p.$$

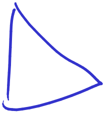
$$a(\hat{\varphi}_i, \hat{\varphi}_j) = \sum_{m=1}^{M} \int_{E_m} \kappa(x) \cdot \nabla \hat{\varphi}_i \nabla \hat{\varphi}_j$$

$$g(\hat{\varphi}_i) = \sum_{m=1}^{M} \int_{E_m} f(x) \hat{\varphi}_i$$

Integrals on the Reference Element

Evaluate

$$\int_E \kappa(\boldsymbol{x}) \nabla_{\boldsymbol{x}} \hat{\varphi}_i(\boldsymbol{x})^T \nabla_{\boldsymbol{x}} \hat{\varphi}_j(\boldsymbol{x}) d\boldsymbol{x}.$$



$$= \int_E \kappa(x) \left(J_T^{-T} \nabla_{\vec r} \hat{\varphi}_i\right)^T \cdot \left(J_T^{-T} \nabla_r \hat{\varphi}_j\right)$$

mat product

$$\overset{P1}{=} \left(J_T^{-T} \nabla_{\vec r} \hat{\varphi}_i\right)^T \cdot \left(J_T^{-T} \nabla_r \hat{\varphi}_j\right) \int_E \kappa(x)$$

$$= \left(J_T^{-T} \nabla_{\vec r} \hat{\varphi}_i\right)^T \cdot \left(J_T^{-T} \nabla_r \hat{\varphi}_j\right) |J_T| \int_{\hat E} \kappa(T(\vec r))$$

And now the RHS functional.

$$\int f(x)\, \hat{\varphi}_i(x) = |J_T| \int_{\hat E} f(T(\vec r))\, \hat{\varphi}_i(\vec r)\, d\vec r.$$

Inhomogeneous Dirichlet BCs

$u(x) = 0 \to u \in H^1_0$?

Handle an inhomogeneous boundary condition $u(\boldsymbol{x}) = \eta(\boldsymbol{x})$ on $\partial\Omega$.

Find a function $u^0 \in H^1(\Omega)$ so that $u^0(x) = \eta(x)$ on $\partial\Omega$,

Define: $\hat{u} = u - u^0$

$\hat{u} \in H^1_0$

?

Weak form: $u = \hat{u} + u^0$

$$a(u,v) = a(\hat{v}, v) + \underbrace{a(u^0, v)}$$

$$a(\hat{u}, v) = \underbrace{g(v) - a(u^0, v)}$$

"lifting argument"