## Today

- FE Approximation

if: $\|u-v_h\| \leq \|u-I_h u\| \leq h^? \cdot \|u\|_{H^k}$

$v_h \in V_h$

- $\Delta u = f$ ?

  Stokes

- DG

## Announcements

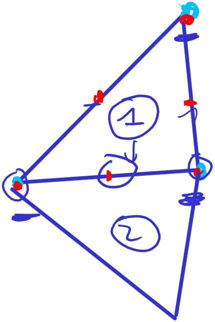
- HW5 now due May 6
  - ↳ P1, P3 Fixed

- Extra credit?

$Ax = b$

$PAx = Pb$

$y = P^T x$




$P^1$

$P^2$

1

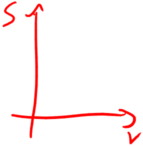
$r$

$s$

$r^2$

$rs$

$s^2$



# Sobolev Seminorms

$$\to \quad \|u\|_{k,p} := \sqrt[p]{\sum_{|\alpha|\leq k} \|D_w^{\alpha} u\|_p^p},$$

$W^{k,p}$

$H^1 = W^{1,2}$

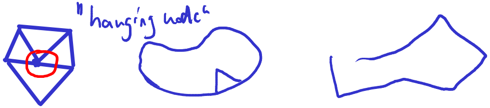
## Definition ((k, p)-Sobolev Seminorm)

The Sobolev seminorm is given by

$$|u|_{k,p} := \sqrt[p]{\sum_{|\vec{\alpha}|=k} \|D_w^{\vec{\alpha}} u\|_p^p}$$

# Conditions on the Mesh

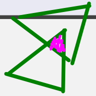
Let $\Omega$ be a polygonal domain.




## Admissibility (Braess, Def. II.5.1)

A partition (mesh) $\mathcal{T} = \{E_1, \ldots, E_M\}$ of $\Omega$ into triangular or quadrilateral elements is called admissible if

- $\bar{\Omega} = \bigcup_{m=1}^{M} \bar{E}_m$



"Nitsche methods"

✗ - If $E_i \cap E_j$ is exactly one point, then that point must be a vertex.

✓ - If $E_i \cap E_j$ is more than a point and $i \neq j$, then $E_i \cap E_j$ must be a common edge

Give an example of a non-admissible partition.

hn, overlap, curvi

# Mesh Resolution, Shape Regularity


## Definition (Diameter)

$$d(E) = \sup\{|x-y| : x, y \in E\}$$

## Mesh Resolution

When every element has a diameter at most $2h$, then we write $\mathcal{T}_h$.

## Definition (Shape Regularity (Braess, Def. II.5.1))

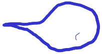
A family of partitions $\{\mathcal{T}_h\}$ is called shape regular if

"incircle crit."

there exists a number $\kappa > 0$ so that every element $E \in \mathcal{T}_h$ contains a circle of radius $\rho_E \geq h/\kappa$.

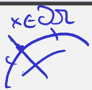
# Cone Conditions

'cusp'



## Definition (Lipschitz Domain)

A bounded domain $\Omega \subset \mathbb{R}^n$ is called a Lipschitz domain provided that...




for every $x \in \partial\Omega$, there exists a nbh so that $\partial\Omega \cap$ nbh is the graph of a Lip. function

Lipschitz domains satisfy a cone condition:


## Theorem (Rellich Selection Theorem (Braess, Thm. II.1.9))

*Let $m \geq 0$, let $\Omega$ be Lipschitz. Then the imbedding $H^{m+1}(\Omega) \to H^m(\Omega)$ is compact, i.e. any bounded sequence in the range of the imbedding has a convergent subsequence.*

# The Interpolation Operator

**Theorem (Interpolation Operator (Braess, Lemma II.6.2))**

*Let $\Omega \subset \mathbb{R}^2$ be Lipschitz. Let $t \geq 2$, and $z_1, z_2, \ldots, z_s$ are $s := t(t+1)/2$ prescribed points in $\bar{\Omega}$ such that the interpolation operator $I : H^t \to \mathbb{P}^{t-1}$ is well-defined. Then there exists a constant $c$ so that for $u \in H^t(\Omega)$*

$$\|u - Iu\|_{H^t} \leq c \, |u|_{H^t}$$

**Theorem (Approx. for Congruent $\triangle$ (Braess, Remark II.6.5))**

*Let $E_h := h\hat{E}$, i.e. a scaled version of a reference triangle, with $h \leq 1$. Then, for $0 \leq m \leq t$, there exists a $C$ so that*

$$\|u - Iu\|_{H^m(E_h)} \leq c \, h^{t-m} \, |u|_{H^t}$$

# Approximation for Congruent Triangles: Proof

$E_h = h\hat{E}$ $\qquad h \le 1$

$$\to \|u - Iu\|_{H^m(E_h)} \le Ch^{t-m}|u|_{H^t(E_h)} \quad (0 \le m \le t)$$

Let $u \in H_t(E_h)$. Define $v \in H^t(\hat{E})$ by $v(y) = u(hy)$.

$D_w^{\vec{\alpha}} v = h^{|\alpha|} D_w^{\vec{\alpha}} u$. $\quad (|\alpha| \le t)$

$\to |v|^2_{H^\ell(\hat{E})} = \sum_{|\alpha|=\ell} \int_{\hat{E}} (D_w^\alpha v)^2 dx = \sum_{|\alpha|=\ell} \int_{E_h} (D^\alpha u)^2 h^{2\ell-2} = |u|^2_{H^\ell} h^{2\ell-2}$

$\|u\|^2_{H^n} = \sum_{\ell \le n} |u|^2_{H^\ell} = \sum_{\ell \le n} h^{-2\ell+2} |v|_{H^\ell} \underset{(h\le 1)}{\le} C' h^{-2m+2} \|v\|^2_{H^n}$.

$\|u - Iu\|_{H^m} \le C' h^{-m+1} \|v - Iv\|_{H^m} \le C' h^{-m+1} \|v - Iv\|_{H^t}$

$\le C'' h^{-m+1} |v|_{H^t} \le C'' h^{t-m} |u|_{H^t}$.

# $H^m$ Polynomial Approximation on Meshes

## Definition (Broken Norm)

Given a partition $\mathcal{T}_h = \{E_i\}_{i=1}^M$ and a function $u$ such that $u \in H^m(E_i)$,

$$\|u\|_{H^m,h} := \sqrt{\sum_{m=1}^{M} \|u\|^2_{H^m(E_m)}}$$

## Approximation Theorem (Braess, Theorem II.6.4)

Let $t \geq 2$, suppose $\mathcal{T}_h$ is a shape-regular triangulation of $\Omega$. Then there exists a constant $c$ such that, for $0 \leq m \leq t$ and $u \in H^t(\Omega)$,

$$\|u - I_h u\|_{H^m,h} \leq c\, h^{t-m}\, |u|_{H^t}$$

# Outline

# Weak Forms as Minimization Problems

$a(u,v) = g(v) \quad \forall v \in V$

Let $V$ be a linear space, and $a : V \times V \to \mathbb{R}$ a bilinear form, and $g \in V'$.

**Theorem (Solutions of Weak Forms are Quadratic Form Minimizers)**

*If $a$ is SPD, then*

$$\boxed{J(v) = \frac{1}{2} a(v,v) - g(v)}$$

*attains its minimum over $V$ at $u$ iff $a(u,v) = g(v)$ for all $v \in V$.* $Ax = b$

$u, v \in V \quad t \in \mathbb{R}$ $\qquad x^T A x$

$$J(u+tv) = \frac{1}{2} a(u+tv, u+tv) - g(u+tv)$$

$$= J(u) + t\,[\underbrace{a(u,v) - g(v)}_{=0}] + \frac{t^2}{2} \underline{a(v,v)}$$

If $u$ satisfies $a(u,v) = g(v)$, $J(u+tv) > J(u)$.

If $J$ has a min at $u$, $\varphi: t \mapsto J(u+tv)$ has a min at $t=0$, $\varphi'(0) = 0$.

# Saddle Point Problems

$X$, $M$ Hilbert spaces. $a : X \times X \to \mathbb{R}$ and $b : X \times M \to \mathbb{R}$ continuous bilinear forms, $f \in X'$, $g \in M'$. Minimize

$$J(u) = \frac{1}{2}a(u,u) - \langle f, u\rangle \qquad \text{subject to} \qquad b(u,\mu) = \langle g, \mu\rangle \quad (\mu \in M).$$

Apply the method of the Lagrange multipliers.

$$\mathcal{L}(u,\lambda) = J(u) + \left[ b(u,\lambda) - (g,\lambda) \right]$$

# Example: Saddle Point Problem in $\mathbb{R}^2$

$$\nabla \mathcal{L} = 0$$

$$\begin{aligned} f(x,y) &= x^2 + y^2 &\to\quad &\text{min!} \\ g(x,y) &= x + y &=\quad &2 \end{aligned}$$

$$\left( \begin{array}{l} -\nabla f = \lambda \nabla g \\ g(x,y) = 0 \end{array} \right.$$

Write down the Lagrangian.

$$\mathcal{L}(x,y,\lambda) = f(x,y) - \lambda g(x,y)$$

Show that $x = y = 1$, $\lambda = -2$ is a saddle point.