Today
- classification
- computing derivatives
- FD for hyperbolic

Announcements
- Office Hours
- HW1
Elliptic PDE: Laplace/Poisson Equation

\[ \triangle u = \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2} = \nabla \cdot \nabla u(x) \stackrel{2D}{=} u_{xx} + u_{yy} = f(x) \quad (x \in \Omega) \]

Called Laplace equation if \( f = 0 \). With Dirichlet boundary condition

\[ u(x) = g(x) \quad (x \in \partial \Omega). \]

**Demo:** Elliptic PDE Illustrating the Maximum Principle
Elliptic PDEs: Singular Solution

$$\Delta w(x) = \int \gamma(x) \varphi(x) \, dx = \varphi(0)$$

**Demo:** Elliptic PDE Radially Symmetric Singular Solution

Given $G(x) = C \log(|x|)$ as the free-space Green’s function, can we construct the solution to the PDE with a more general $f$?

$$u(x) = G \ast f(x) = \int G(x-y) f(y) \, dy$$

What can we learn from this?

Solutions to elliptic PDEs have global dependence on input data.

$$\Delta u = f \quad \Delta g = f$$
Elliptic PDEs: Justifying the Singular Solution

\[ (\rho \ast g)' = (\rho' \ast g) = (\rho \ast g') \]

\[ u(x) = (G \ast f)(x) = \int_{\mathbb{R}^d} G(x - y)f(y)dy \]

Why?

\[ \Delta u(x) = \Delta (G \ast f)(x) = \left( \Delta G \ast f \right)(x) = \int_{\mathbb{R}^d} \Delta G(x - y)f(y)dy = f(x) \]
Parabolic PDE: Heat Equation

\begin{align*}
V'(t) \cdot u(x) &= V(t) \cdot u''(x) \\
&\quad \left( \frac{V'(t)}{V(t)} = c = \frac{u''(x)}{u(x)} \right)
\end{align*}

\[ u_t = u_{xx} \quad ((x, t) \in [0, 1] \times [0, T]) \]

\[ u(x, 0) = g(x) \quad (x \in [0, 1]) \]

\[ u(0, t) = u(1, t) = 0 \quad (t \in [0, T]) \]

with \( g(x) = \sin(\pi x) \). Separation-of-variables analytic solution?

\[
\begin{align*}
V(t) &= e^{-\frac{n^2 \pi^2 t}{T}} \\
W(x) &= \sum_{n=1}^{\infty} A_n \sin(n \pi x) \\
W(0) &= W(1) = 0 \\
W'(1) &= 2 \sin(n \pi) \\
\end{align*}
\]
Parabolic PDE: Solution Behavior

**Demo:** Parabolic PDE What can we learn from analytic and numerical solution?

- Washes out solution
- Appears to satisfy a maximum principle
- Appears to smooth the input data
Hyperbolic PDE: Wave Equation

\[ u_{tt} = c^2 u_{xx} \quad ((x, t) \in \mathbb{R} \times [0, T]) \]

\[ u(x, 0) = g(x) \quad (x \in \mathbb{R}) \]

with \( g(x) = \sin(\pi x) \).

Is this problem well-posed?

\[ u_t(x, 0) = 0 \quad (x \in \mathbb{R}) \]

Can be rewritten in conservation law form:

\[ \frac{\partial q}{\partial t} + \nabla \cdot f(q) = s(x) \]
Hyperbolic Conservation Laws

\[ \frac{\partial q}{\partial t} + \nabla \cdot F(q(x, t)) = s(x) \]

Why is this called a conservation law?

- Balance of conserved quantity \( q \) and a flux \( F \)
- Flux function \( F \) prescribes the direction of flow
- Integral form shows "flow across boundary"

\[ F : \mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{R}^d \]
Wave Equation as a Conservation Law

\[ u_{tt} = c^2 u_{xx} \]

Rewrite the wave equation in conservation law form:

Introduce a new variable:

\[ u_t = c v_x \]
\[ v_t = c u_x \]

\[ u_{tt} = c v_{xt} = c^2 u_{xx}. \]
Solving Conservation Laws

Solve

\[ q = \begin{pmatrix} u \\ v \end{pmatrix} \]

\[ u_t = c v_x \]

\[ v_t = c u_x. \]

\[ q_t - \nabla \cdot \left( \begin{pmatrix} 0 & -c \\ -c & 0 \end{pmatrix} \right) \begin{pmatrix} u \\ v \end{pmatrix} = 0 \]

\[ \hat{q} = \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} \]

Eigenvalues: \( c, -c \)

\[ \hat{q}_t + v^{-1} A \hat{V} q_x = 0 \]

\[ \hat{u}_t = -c \hat{u}_x \]

\[ \hat{v}_t = c \hat{v}_x \]

Solution: \( u(x,t) = \phi(u)(x - ct) + \phi(v)(x - ct) \)

Demo: Hyperbolic PDE
Properties of the solution for hyperbolic equations:

- Models conserved quantities.
- Energies (conserved) \((\rightarrow HU)\).
- Maintain the smoothness of the IC.
- Characteristic decomposition; popular trick.
Outline

Introduction
  Notes
  Notes (unfilled, with empty boxes)
  About the Class
  Classification of PDEs
  Preliminaries: Differencing
  Interpolation Error Estimates (reference)

Finite Difference Methods for Hyperbolic Problems

Finite Volume Methods for Hyperbolic Conservation Laws

Finite Difference Methods for Elliptic Problems

Finite Element Methods for Elliptic Problems

Discontinuous Galerkin Methods for Hyperbolic Problems

A Glimpse of Integral Equation Methods for Elliptic Problems
Interpolation and Vandermonde Matrices

\[ V \overrightarrow{a} = \text{point values} \]

\[ \rho(x_i) = \sum_{j=0}^{\infty} a_j \varphi_j(x_i) \]

generalized Vandermonde matrix

\[ \rho'(x_i) \approx \left( \sum_{j=0}^{\infty} a_j \varphi'_j(x_i) \right) \rightarrow V^{\prime \prime} \rho'(x_i) \]

\[ \rho'(x) \approx \left( V' V^{-1} \rho(x) \right) \]

\[ V'_{i:j} \]
n points:

Interpolation: \( h^n \)

Differentiation: \( h^{n-1} \)
Taking Derivatives Numerically

Why *shouldn’t* you take derivatives numerically?

**Demo:** Taking Derivatives with Vandermonde Matrices