Numerical Methods for Partial Differential Equations

CS555 / MATH555 / CSE510

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Spring 2020
Outline

Introduction
  Notes
  Notes (unfilled, with empty boxes)
  About the Class
  Classification of PDEs
  Preliminaries: Differencing
  Interpolation Error Estimates (reference)

Finite Difference Methods for Hyperbolic Problems

Finite Volume Methods for Hyperbolic Conservation Laws

Finite Difference Methods for Elliptic Problems

Finite Element Methods for Elliptic Problems

Discontinuous Galerkin Methods for Hyperbolic Problems

A Glimpse of Integral Equation Methods for Elliptic Problems
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What’s the point of this class?

PDEs describe lots of things in nature:

- Fluid flow
- Electromagnetics
- Advection
- Plasma
- Mechanics

Idea: Use them to

- make predictions and check them/ check models
- use predictions to answer design questions
Survey

- Home dept
- Degree pursued
  - Longest program ever written
    - in Python?
- Research area
Class web page


- Book Draft
- Notes, Class Outline
- Assignments (submission and return)
- Piazza
- Grading Policies/Syllabus
- Video
- Scribbles
- Demos (binder)
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PDEs: Example 1

What does this do? \( \partial_t u = \partial_x u \)

- slope in \( x \) and \( t \) is identical
- maybe a profile propagated on a display
PDEs: Example II

\[ \partial_x^2 u + \partial_y^2 u = 0 \]

What does this do? 

- "boudiness in \(x\) = -" in \(y\)
- maybe no interior maxima
Some good questions

- What is a time-like variable? (Variables labeled $t$?)
- What if there are boundaries?
  - In space?
  - In time?
- Existence and Uniqueness of Solutions?
  - Depends on where we look (the function space)
  - In the case of the two examples? (if there are no boundaries?)

Some general takeaways:

- Use intuition, it helps
Looking for $u : \Omega \to \mathbb{R}^n$ where $\Omega \subseteq \mathbb{R}^d$ so that $u \in V$ and

$$F(u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \ldots, x, y, \ldots) = 0$$

**Notation**

Used as convenient:

$$u_x = \partial_x u = \frac{\partial u}{\partial x}$$
Properties of PDEs

What is the **order** of the PDE?

- highest derivative taken

When is the PDE **linear**?

- have \(u, v\) solutions \(\alpha u + \beta v\) also a sol.

When is the PDE **quasilinear**?

- The dependence on the highest-order derivative is linear in \(u\).

When is the PDE **semilinear**?

- It is quasilinear and if the h.o.c. only depend on space.
Examples: Order, Linearity?

\[(xu^2)u_{xx} + (u_x + y)u_{yy} + u^3_x + yu_y = f\]

QL \[\neq \partial u\]

\[(x + y + z)u_x + (z^2)u_y + (\sin x)u_z = f\]

SL \[\neq \text{ls}\]
Properties of Domains

$S \subseteq \mathbb{R}^n$

- has corner
- convex / has no true corner
- satisfies minimum angle reg.

May influence existence/uniqueness of solutions!
Function Spaces: Examples

Name some function spaces with their norms.

- \( C^0 \) : continuous
- \( C^1 \)
- \( C^\infty \) : complete continuous
- \( L^p(\Omega) \) \( \| f \|_p = \left( \int_\Omega |f(x)|^p \, dx \right)^{1/p} \)
  \( p = 2 \): inner product/define equivalent classes
- \( W^{1,p}(\Omega) \) \( \| f \|_{W^{1,p}} = \| f \|_p + \| f' \|_p, \| \cdot \|_p < \infty \)

May also influence existence/uniqueness of solutions!

\( H^1 = W^{1,2}_2 \) Hilbert; complete inner prod.
Solving PDEs

Closed-form solutions:

- If separation of variables applies to the domain: good luck with your ODE
- If not: Good luck! → Numerics

General Idea (that we will follow some of the time)

- Pick $V_h \subseteq V$ finite-dimensional
  - $h$ is often a mesh spacing
- Approximate $u$ through $u_h \in V_h$
- Show: $u_h \rightarrow u$ (in some sense) as $h \rightarrow 0$
About grand big unifying theories

Is there a grand big unifying theory of PDEs?

no
Collect some stamps

\[ a(x, y)u_{xx} + 2b(x, y)u_{xy} + c(x, y)u_{yy} + d(x, y)u_x + e(x, y)u_y + f(x, y)u = g(x, y) \]

<table>
<thead>
<tr>
<th>Discriminant value</th>
<th>Kind</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b^2 - 4ac &lt; 0 )</td>
<td>Elliptic</td>
<td>Laplace ( u_{xx} + u_{yy} = 0 )</td>
</tr>
<tr>
<td>( b^2 - 4ac = 0 )</td>
<td>Parabolic</td>
<td>Heat ( u_t = u_{xx} )</td>
</tr>
<tr>
<td>( b^2 - 4ac &gt; 0 )</td>
<td>Hyperbolic</td>
<td>Wave ( u_{tt} = u_{xx} )</td>
</tr>
</tbody>
</table>

Where do these names come from?

**quadratic forms**
Classification in higher dimensions

\[ Lu := \sum_{i=1}^{d} \sum_{j=1}^{d} a_{i,j}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \text{lower order terms} \]

Consider the matrix \( A(x) = (a_{ij}(x))_{i,j} \). May assume \( A \) symmetric. Why?

Schwarz's thm.

What cases can arise for the eigenvalues?

Real eigenvalues
\[ \lambda = 0 \rightarrow \text{parabolic} \]

for some \( \lambda \)

\[ \forall (x) \text{ all have same sign} \]

\[ \forall (x) \quad \Rightarrow \quad \text{elliptic} \]

\[ \forall (x) \text{ all but one } \rightarrow \text{hyperbolic} \]

more than one of each sign

\[ \rightarrow \quad \text{ultra-hyperbolic} \]
Elliptic PDE: Laplace/Poisson Equation

\[ \Delta u = \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2} = \nabla \cdot \nabla u(x) \equiv u_{xx} + u_{yy} = f(x) \quad (x \in \Omega) \]

Called Laplace equation if \( f = 0 \). With Dirichlet boundary condition

\[ u(x) = g(x) \quad (x \in \partial \Omega). \]

**Demo:** Elliptic PDE Illustrating the Maximum Principle

\[ g(x) = \sin(\alpha) \]

\[ u(y, \theta) \]

\[ u(v \cdot U(\theta)) \]
Elliptic PDEs: Singular Solution

**Demo:** Elliptic PDE Radially Symmetric Singular Solution

Given \( G(x) = C \log(|x|) \) as the free-space Green’s function, can we construct the solution to the PDE with a more general \( f \)?

What can we learn from this?