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Harten’s Lemma

**Theorem (Harten’s Lemma)**

If a scheme can be written as

\[ \tilde{u}_{j,\ell+1} = \tilde{u}_{j,\ell} + \lambda \left( \frac{C_{j+1/2}}{2} \Delta_+ \tilde{u}_j - \frac{D_{j-1/2}}{2} \Delta_- \tilde{u}_j \right) \]

with \( C_{j+1/2} \geq 0, \ D_{j+1/2} \geq 0, \ 1 - \lambda (C_{j+1/2} + D_{j+1/2}) \geq 0 \) and \( \lambda = \Delta t / \Delta x \), then it is TVD.

As a matter of notation, we have

\[ \Delta_+ u_j = u_{j+1} - u_j, \]
\[ \Delta_- u_j = u_j - u_{j-1}. \]

We have omitted the time subscript for the time level \( \ell \).
Minmod Scheme

Still assume \( f'(u) \geq 0 \).

\[
\begin{align*}
\bar u_{j+1/2}^{(1)} &= f\left( \bar u_j + \frac{1}{2}(\bar u_{j+1} - \bar u_j) \right), \\
\bar u_{j+1/2}^{(2)} &= f\left( \bar u_j + \frac{1}{2}(\bar u_j - \bar u_{j-1}) \right).
\end{align*}
\]

Design a ‘safe’ thing to use for \( \tilde u \):

\[
\tilde u_j = \min\max\left( u_j, \bar u_j^{(1)}, \bar u_j^{(2)} \right)
\]

\[
f_{j+1/2} = f\left( \tilde u_j + \tilde u_j \right)
\]
Minmod is TVD

$$\tilde{u}_{j+1} = \tilde{u}_j + \lambda \left( C \Delta_+ - D \Delta_- \right)$$

Show that Minmod is TVD:

$$\tilde{u}_{j+1} = \tilde{u}_j - \lambda \left( f(\tilde{u}_j + \tilde{u}_{j-1}) - f(\tilde{u}_{j-1} + \tilde{u}_{j-2}) \right)$$

$$= \tilde{u}_j - \lambda \left( D_{j+\frac{1}{2}} \Delta_- \tilde{u}_j \right)$$

$$D_{j+\frac{1}{2}} = \frac{f(\tilde{u}_j + \tilde{u}_{j-1}) - f(\tilde{u}_{j-1} + \tilde{u}_{j-2})}{\tilde{u}_j - \tilde{u}_{j-1}} = f'(\tilde{\xi}) \frac{\tilde{u}_j - \tilde{u}_{j-1} + \tilde{u}_{j-1} - \tilde{u}_{j-2}}{\tilde{u}_j - \tilde{u}_{j-1}}$$

$$= \frac{f'(\tilde{\xi})}{\tilde{u}_j - \tilde{u}_{j-1}} \left[ 1 + \frac{\tilde{u}_j - \tilde{u}_{j-1}}{\tilde{u}_j - \tilde{u}_{j-1}} - \frac{\tilde{u}_{j-1}}{\tilde{u}_j - \tilde{u}_{j-1}} \right] \geq 0$$
Minmod: CFL restriction?

Derive a time step restriction for Minmod.

\[ D_{j-\frac{1}{2}} \leq \varphi'(q_j) \cdot \frac{3}{2} \leq \max |\varphi'(q)| \cdot \frac{3}{2} \]

\[ 0 \leq 1 - \lambda D_{j-\frac{1}{2}} \geq 1 - \frac{3}{2} \lambda \max |\varphi'(q)| \leq \lambda \max |\varphi'(q)| \leq \frac{\lambda}{3} \]
What about Time Integration?  

\[ u^{(1)} = u_{\ell} + h_t L(u_{\ell}), \quad u_{\ell+1} = \frac{u_{\ell}}{2} + \frac{1}{2} (u^{(1)} + h_t L(u^{(1)})). \]

Above: A version of RK2 with \( L \) the ODE RHS. Will this cause wrinkles?

\[
- \quad TV\left( \alpha \frac{\tilde{u} + (1-\alpha) \tilde{v}}{2} \right) \leq \alpha TV(\tilde{u}) + (1-\alpha) TV(\tilde{v}) \quad (0 \leq \alpha \leq 1)
\]

\[
TV\left( u_{\ell+1} \right) = TV\left( \frac{u_{\ell}}{2} + \frac{1}{2} \left( u^{(1)} + h_t L(u^{(1)}) \right) \right)
\leq \frac{1}{2} TV(u_{\ell}) + \frac{1}{2} TV\left( u^{(1)} + h_t L(u^{(1)}) \right)
\leq \frac{1}{2} TV(u_{\ell}) + \frac{1}{2} TV\left( u_{\ell} + h_t L(u_{\ell}) \right)
\leq \frac{1}{2} TV(u_{\ell}) + \frac{1}{2} TV\left( u_{\ell} \right) = TV(u_{\ell})
\]
Total Variation is Convex

Show: $\text{TV}(\cdot)$ is a convex functional.

\[
0 \leq \alpha \leq 1 \\
\text{TV}(\alpha \tilde{u} + (1-\alpha) \tilde{v}) \\
\leq \sum_j \left| \alpha \left( u_j - u_{j-1} \right) + (1-\alpha) \left( v_j - v_{j-1} \right) \right| \\
\leq \alpha \sum_j |u_j - u_{j-1}| + (1-\alpha) \sum_j |v_j - v_{j-1}| \\
\leq \alpha \text{TV}(u) + (1-\alpha) \text{TV}(v)
\]
Can TVD schemes be high order everywhere?

Near smooth extrema, TVD schemes are still only first order.
High Order at Smooth Extrema

- TVB Schemes [Shu '87]
- ENO [Harten/Engquist/Osher/Chakravarthy '87]
  - Define \( W_j = w(x_{j+1/2}) = \int_{x_{1/2}}^{x_{j+1/2}} u(\xi, t) d\xi = h_x \sum_{i=1}^{j} \bar{u}_i \)
  - Observe \( u_{j+1/2} = w'(x_{j+1/2}) \).
  - Approximate by interpolation/numerical differentiation.
  - Start with the linear function \( p^{(1)} \) through \( W_{j-1} \) and \( W_j \)
  - Compute divided differences on \( (W_{j-2}, W_{j-1}, W_j) \)
  - Compute divided differences on \( (W_{j-1}, W_j, W_{j+1}) \)
  - Use the one with the smaller magnitude to extend \( p^{(1)} \) to quadratic
  - (and so on, adding points on the side with the lowest magnitude)

- WENO [Liu/Osher/Chan '94]
Outline

Introduction

Finite Difference Methods for Time-Dependent Problems

Finite Volume Methods for Hyperbolic Conservation Laws
- Theory of 1D Scalar Conservation Laws
- Numerical Methods for Conservation Laws
- Higher-Order Finite Volume
- Outlook: Systems and Multiple Dimensions

Finite Element Methods for Elliptic Problems

Discontinuous Galerkin Methods for Hyperbolic Problems

A Glimpse of Integral Equation Methods for Elliptic Problems
Systems of Conservation Laws

Linear system of hyperbolic conservation laws, $A \in \mathbb{R}^{m \times m}$:

$$u_t + Au_x = 0,$$

$$u(x,0) = u_0(x).$$

Assumptions on $A$?

Hyperbolic if $A$ diagonalizable, real eigenvalues

$$A \tilde{r}_p = \lambda_p \tilde{r}_p \quad (p = 1 \ldots m)$$

Called strictly hyperbolic if $\lambda_p$ are all distinct.

$$v = R^{-1} \tilde{u} \quad \Rightarrow \quad v_t + \Lambda v_x = 0$$

$$AR = R \Lambda$$

"Characteristic variables"
Linear System Solution

\[ \mathbf{v} = R^{-1} \mathbf{u}, \quad \mathbf{v}_t + \Lambda \mathbf{v}_x = 0. \]

Write down the solution.

\[ w(x, t) = \sum_{\rho} r_{\rho} v_{\rho}(x - \lambda_{\rho} t) \]

\[ \mathbf{v}(x) = R^{-1} \mathbf{w}(x, 0) \]

What is the impact on boundary conditions? E.g. \( (\lambda_{\rho}) = (\text{c}, 0, \text{c}) \) for a BC at \( x = 0 \) for \([0, 1]\)?

\[ \text{Cannot specify a BC on } v_3. \]
Characteristics for Systems (1/2)

Consider system $u_t + f(u)_x = 0$. Write in quasilinear form:

$$u_t + A(u) u_x = 0 \quad A(u) = f'(u)$$

When hyperbolic?

$$A(u) \text{ diagonal \ w/ real eig.} \quad \text{strictly if eigv. are distinct. \ only local}$$
What about characteristics/shock speeds?

- Considering char. var.: "characteristic" still makes sense in characteristics through \((x, t)\)
- Char. speeds: not on ODE

Are values of \(u\) still constant along characteristics?

no, but CV are.
Shocks and Riemann Problems for Systems

\[ u_t + Au_x = 0, \]
\[ u(x, 0) = \begin{cases} u_l & x < 0, \\ u_r & x > 0. \end{cases} \]

Solution? (Assume strict hyperbolicity with \( \lambda_1 < \lambda_2 < \cdots < \lambda_m \).)