Today
- $\Delta u = f$
- Poincaré's ineq.
- $a(u, v) = b(v)$ $\forall v \in V$
- $L^2 a(u_h, v_h) = b(v_h)$ $\forall v_h \in V_h$
- $\|u - u_h\|_{H^1} \leq \text{Céa's Lemma}$
- $\|u - u_h\|_{L^2} \leq \text{Aubin-Nitsche Lemma}$

Announcements
- HW deadline $\rightarrow$ Apr. 15 (next Wed)
- proj. deadline $\rightarrow$ April 22
- assignment for rest of semester
- Firedrake install
Poisson

Let \( \Omega \subset \mathbb{R}^n \) open, bounded, \( f \in H^{-1}(\Omega) \).

\[- \nabla \cdot \nabla u = f \]
\[u(x) = 0 \quad \text{on \( \partial \Omega \)}\]

This is called the **Poisson problem** (with Dirichlet BCs).

Weak form?

\[
\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f(x)v(x) \, dx \quad \forall v \in H^1_0
\]

\[
\begin{align*}
& \quad \quad a(u,v) \quad \quad g(v) \\
& \quad \quad u \in H^1_0
\end{align*}
\]
Ellipticity

Let $V$ be a Hilbert space.

**V-Ellipticity**

A bilinear form $a(\cdot, \cdot) : V \times V \to \mathbb{R}$ is called **coercive** if there exists a constant $c_0 > 0$ so that

$$c_0 \|u\|_V^2 \leq a(u, u)$$

and $a$ is called **continuous** if there exists a constant $c_1 > 0$ so that

$$|a(u, v)| \leq c_1 \|u\|_V \|v\|_V$$

If $a$ is both coercive and continuous on $V$, then $a$ is said to be $V$-elliptic.
Lax-Milgram Theorem

Let \( V \) be Hilbert space with inner product \( \langle \cdot, \cdot \rangle \).

\[ a(u, v) = a(v, u) \]

Lax-Milgram, Symmetric Case

Let \( a \) be a \( V \)-elliptic bilinear form that is also symmetric, and let \( g \) be a bounded linear functional on \( V \).

Then there exists a unique \( u \in V \) so that \( a(u, v) = g(v) \) for all \( v \in V \).
Back to Poisson

Can we declare victory for Poisson?

\[
\left| \int_S \nabla u \cdot \nabla v \, dx \right| = \frac{1}{2} \left \| \nabla u \right \|_2 \left \| \nabla v \right \|_2 \leq \left \| \nabla u \right \|_2 \left \| v \right \|_2 \\
\int_S \nabla u \cdot \nabla u \, dx \geq \varepsilon_0 \left( \int_S \nabla u \cdot \nabla u + \nabla^2 u \right) \\
\| \nabla u \|_2 \leq C \cdot \| u \|_2
\]

Can this inequality hold in general, without further assumptions?

Constant violates that \( \varepsilon_0 \) is such.
Theorem (Poincaré-Friedrichs Inequality)

Suppose $\Omega \subset \mathbb{R}^n$ is bounded and $u \in H^1_0(\Omega)$. Then there exists a constant $C > 0$ such that

$$\|u\|_{L^2} \leq C \|\nabla u\|_{L^2}.$$
Prove the result in $C_0^\infty(\Omega)$. 

\[ \|u\|_2^2 = \int_{\Omega} \frac{1}{n} \cdot \nabla \cdot (u^2 x) - \frac{2}{n} \cdot u \cdot (\nabla u \cdot x) \, dx \]
\[ = \frac{1}{n} \int_{\Omega} \nabla \cdot (u^2 x) \, dS_x - \frac{2}{n} \int_{\Omega} u \cdot (\nabla u \cdot x) \, dx \]
\[ \leq \frac{2}{n} \max_{x \in \partial \Omega} |x|_2 \int_{\Omega} |u| \, dx \leq \frac{2}{n} \max_{x \in \partial \Omega} \|u\| \cdot \|u\|_2^2 \]

\[ \Rightarrow \|u\|_2 \leq C \cdot \|\nabla u\|_2 \]
Prove the result in $H^1_0(\Omega)$.

Let $u \in H^1_0(\Omega)$, $(u_k) \subset C^\infty(\Omega)$ so that $\|u_k - u\|_{H^1} \to 0$.

Then the inequality holds for each $u_k$.

By continuity, the inequality also holds for $u$. 
Show that the Poisson bilinear form is coercive.

\[
\frac{1}{c^2+1} \|u\|_{H^1}^2 \leq \frac{1}{c^2+1} \left( \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \right) \leq c^2 \|\nabla u\|_{L^2}^2 = a(u, u).
\]

Draw a conclusion on Poisson:

Because of coercivity and continuity, Poisson has a unique solution in \( H_0^1(\Omega) \).
Outline

Introduction

Finite Difference Methods for Time-Dependent Problems

Finite Volume Methods for Hyperbolic Conservation Laws

Finite Element Methods for Elliptic Problems
  tl;dr: Functional Analysis
  Back to Elliptic PDEs
  Galerkin Approximation
  Finite Elements: A 1D Cartoon
  Finite Elements in 2D
  Non-symmetric Bilinear Forms
  Mixed Finite Elements

Discontinuous Galerkin Methods for Hyperbolic Problems

A Glimpse of Integral Equation Methods for Elliptic Problems
Ritz-Galerkin

Some key goals for this section:

- How do we use the weak form to compute an approximate solution?
- What can we know about the accuracy of the approximate solution?

Can we pick one underlying principle for the construction of the approximation?

\[ a(u, v) = g(v) \quad \forall v \in V \quad V \subseteq H \]

Choose a finite-dim subspace \( V_h \subseteq V \). Find a solution \( u_h \in V_h \)

\[ a(u_h, v) = g(v) \quad \forall v \in V_h \]

\( u_h \) is called the Ritz-Galerkin approximation.
Galerkin Orthogonality

\[ a(u, v) = g(v) \quad \text{for all } v \in V, \ a(u_h, v_h) = g(v_h) \quad \text{for all } v_h \in V_h. \]

Observations?

\[ a(u, v_h) = g(v_h) \quad v_h \in V_h \]

\[ \Rightarrow a(u - u_h, v_h) = 0 \]

approx error

Galerkin orthogonality
Céa’s Lemma

Let $V \subset H$ be a closed subspace of a Hilbert space $H$.

Let $a(\cdot, \cdot)$ be a coercive and continuous bilinear form on $V$. In addition, for a bounded linear functional $g$ on $V$, let $u \in V$ satisfy

$$a(u, v) = g(v) \quad \text{for all } v \in V.$$  

Consider the finite-dimensional subspace $V_h \subset V$ and $u_h \in V_h$ that satisfies

$$a(u_h, v_h) = g(v_h) \quad \text{for all } v_h \in V_h.$$  

Then

$$\| u - u_h \|_V \leq \frac{c}{c_0} \inf_{v_h \in V_h} \| u - v_h \|.$$
Recall Galerkin orthgonality: $a(u_h - u, v_h) = 0$ for all $v_h \in V_h$. Show the result.

$$c_2 \| u - u_h \|_V^2 \leq a(u - u_h, u - u_h) \quad \text{(coercivity)} \quad \forall v_h \in V_h$$

$$= a(u - u_h, u - v_h) + a(u - u_h, v_h - u_h)$$

$$\leq a(u - u_h, u - v_h) \leq c_1 \| u - u_h \|_V \| u - v_h \|_V.$$
**Elliptic Regularity**

### Definition (\(H^s\) Regularity)

Let \( m \geq 1 \), \( H^m_0(\Omega) \subseteq V \subseteq H^m(\Omega) \) and \( a(\cdot, \cdot) \) a \( V \)-elliptic bilinear form. The bilinear form \( a(u, v) = \langle f, v \rangle \) for all \( v \in V \) is called \( H^s \) regular, if for every \( f \in H^{s-2m} \) there exists a solution \( u \in H^s(\Omega) \) and we have with a constant \( C(\Omega, a, s) \),

\[
\|u\|_{H^s} \leq \frac{\|f\|_{H^{s-2m}}}{C(\Omega, a, s)}
\]

### Theorem (Elliptic Regularity (cf. Braess Thm. 7.2))

Let \( a \) be a \( H^1_0 \)-elliptic bilinear form with sufficiently smooth coefficient functions.

- If \( \Omega \) is convex, then the Dirichlet problem is \( H^s \) regular.
- Let \( s \geq 2 \). If \( \Omega \) is \( C^1 \), then the \( \partial \Omega \) Dirichlet problem is \( H^s \) regular.
Elliptic Regularity: Counterexamples

Are the conditions on the boundary essential for elliptic regularity?

Are there any particular concerns for mixed boundary conditions?
Estimating the Error in the Energy Norm

Come up with an idea of a bound on $\|u - u_h\|_{H^1}$.

\[
\|u - u_h\|_{H^1} \leq C \inf_{v \in V_h} \|u - v\|_{H^1} \leq C \|u - \bar{u}_h n\|_{H^1} \\
\leq C' h \|u\|_{H^2} \leq C' h \cdot C(\mathcal{O}_h \mathcal{S}) \|f\|_{L^2}
\]

What’s still to do?