Today

- $L^2$ estimate
  - Aubin - Nitsche
  - Elliptic regularity
- FEM assembly 1D
- FEM 2D
- FEM approximation
- Mixed FEM

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Announcements

- HW4 due
- HW5 out soon
- Project due in a week

\[ \| u - u_h \|_{H^1} \leq C \inf_{v \in V_h} \| u - v \|_{H^1} \leq C \]
Céa’s Lemma

Let $V \subset H$ be a closed subspace of a Hilbert space $H$.

Céa’s Lemma

Let $a(\cdot, \cdot)$ be a coercive and continuous bilinear form on $V$. In addition, for a bounded linear functional $g$ on $V$, let $u \in V$ satisfy

$$a(u, v) = g(v) \quad \text{for all } v \in V.$$ 

Consider the finite-dimensional subspace $V_h \subset V$ and $u_h \in V_h$ that satisfies

$$a(u_h, v_h) = g(v_h) \quad \text{for all } v_h \in V_h.$$ 

Then

$$\|u - u_h\| \leq \frac{c_1}{c_0} \inf_{v_h \in V_h} \|u - v_h\|.$$
Elliptic Regularity

Definition ($H^s$ Regularity)

Let $m \geq 1$, $H^m_0(\Omega) \subseteq V \subseteq H^m(\Omega)$ and $a(\cdot, \cdot)$ a $V$-elliptic bilinear form. The bilinear form $a(u, v) = \langle f, v \rangle$ for all $v \in V$ is called $H^s$ regular, if for every $f \in H^{s-2m}$ there exists a solution $u \in H^s(\Omega)$ and we have with a constant $C(\Omega, a, s)$,

$$\| u \|_{H^s} \leq C(\Omega, a, s) \| f \|_{H^{s-2m}}$$

Theorem (Elliptic Regularity (cf. Braess Thm. 7.2))

Let $a$ be a $H^1_0$-elliptic bilinear form with sufficiently smooth coefficient functions.

- $\Omega$ convex $\Rightarrow$ $H^2$-regular (Dirichlet)

$-s \geq 2 \Rightarrow \Omega \text{ C^s} \Rightarrow$ Dirichlet prob is $H^s$-regular
Elliptic Regularity: Counterexamples

Are the conditions on the boundary essential for elliptic regularity?

\[ \Delta u = 0 \quad \text{on circ} \quad u = 0 \text{ on } \gamma \]

\[ u(z) = 1 + \frac{1}{z^{2/3}} \]

\[ u'(z) = \frac{2}{3} z^{-1/3} \]

Are there any particular concerns for mixed boundary conditions?

mixed BC problem

'eqvi.' to pure Dirichlet on bigger domain with relevant cone

\[ \text{Gilbarg / Trudinger} \]
Estimating the Error in the Energy Norm

Come up with an idea of a bound on $\|u - u_h\|_{H^1}$.

$$\|u - u_h\|_{H^1} \leq \frac{C_1}{C_0} \inf_{v_h \in V_h} \|v_h - u\|_{H^1} \leq C \|u - I_h u\|_{H^1} \leq \frac{C}{h^2} \|u\|_{H^2} \leq C_2 h \|u\|_{L^2}$$

What’s still to do?

- $V_h$?
- $I_h$?
- interp. error bound
- $L^2$ error
\( L^2 \) Estimates

Let \( H \) be a Hilbert space with the norm \( \| \cdot \|_H \) and the inner product \( \langle \cdot , \cdot \rangle \). (Think: \( H = L^2 \), \( V = H^1 \).)

Theorem (Aubin-Nitsche)

Let \( V \subseteq H \) be a subspace that becomes a Hilbert space under the norm \( \| \cdot \|_V \). Let the embedding \( V \to H \) be continuous. Then we have for the finite element solution \( u \in V_h \subset V \):

\[
\| u - u_h \|_V \leq c_1 \| u - u_h \|_V \cdot \sup_{g \in H} \left[ \frac{1}{\| g \|_H} \inf_{v_h \in V_h} \| \phi_g - v_h \|_V \right]
\]

if with every \( g \in H \) we associate the unique (weak) solution \( \phi_g \) of the equation (also called the dual problem)

\[
\langle a(u, v) = \langle g, v \rangle \text{ for all } v \in V
\]
\[ a(n, v) = \sum_{x} D_n(x) \cdot D_v(x) = \sum_{x} f_n(x) \cdot f_v(x) \]

Theorem 1
Aubin-Nitsche: Proof

\[ \| u - u_h \|_H = \sup_{g \in H} \frac{\langle g, u - u_h \rangle}{\| g \|_H} \]

\[ \langle g, u - u_h \rangle = \langle g, u - u_h \rangle_{\text{dual}} = a(u - u_h, \varphi_g) = a(u - u_h, \varphi_g - u_h) \]

\[ \leq C_{\text{cont.}} \| u - u_h \|_V \| \varphi_g - u_h \|_V \]

\[ \langle g, u - u_h \rangle \leq C_r \inf_{v_h \in V_h} \| \varphi_g - u_h \|_V \]

\[ \| u - u_h \|_H = \sup_{g \in H} \frac{\langle g, u - u_h \rangle}{\| g \|_H} \]

\[ \text{Let } u_h \in V_h \]
Estimates using Aubin-Nitsche

\[ \|u - u_h\|_H \leq c_1 \|u - u_h\|_{V} \sup_{g \in H} \left[ \frac{1}{\|g\|_H} \inf_{v_h \in V_h} \|\varphi_g - v_h\|_V \right], \]

If \( u \in H^1_0(\Omega) \), what do we get from Aubin-Nitsche?

\[ \|u - u_h\|_{L^2} \leq C \cdot h \cdot \|u - u_h\|_{H^1} \]

So does Aubin-Nitsche give us an \( L^2 \) estimate?

\[ \|u - u_h\|_{L^2} \leq C \cdot h \cdot h \cdot \|f\|_{L^2} \]
Outline

Introduction

Finite Difference Methods for Time-Dependent Problems

Finite Volume Methods for Hyperbolic Conservation Laws

**Finite Element Methods for Elliptic Problems**
- tl;dr: Functional Analysis
- Back to Elliptic PDEs
- Galerkin Approximation
  - Finite Elements: A 1D Cartoon
  - Finite Elements in 2D
  - Non-symmetric Bilinear Forms
  - Mixed Finite Elements

Discontinuous Galerkin Methods for Hyperbolic Problems

A Glimpse of Integral Equation Methods for Elliptic Problems
Finite Elements in 1D: Discrete Form

Ω := [α, β]. Look for $u \in H^1_0(\Omega)$, so that $a(u, \varphi) = \langle f, \varphi \rangle$ for all $\varphi \in H^1_0(\Omega)$. Choose $V_h = \text{span}\{\psi_1, \ldots, \psi_n\}$ and expand $u_h = \sum_{i=1}^{n} u_h^i \psi_i \in V_h$. Find the discrete system.

$-u'' = f$

$u = h(h)$

\[
\begin{align*}
\sum_{i=1}^{n} a(u_h^i, \varphi) &= \langle f, \varphi \rangle \quad \forall \varphi \in V_h \\
\sum_{i=1}^{n} (\sum_{j=1}^{n} a(\psi_i, \psi_j)) u_h^i &= \langle f, \psi_j \rangle \\
\sum_{i=1}^{n} u_h^i a(\psi_i, \psi_j) &= \langle f, \psi_j \rangle, \quad j = 1, \ldots, n
\end{align*}
\]
Grids and Hats

Let $l_i := [\alpha_i, \beta_i]$, so that $\bar{\Omega} = \bigcup_{i=0}^{N} l_i$ and $l_i \cap l_j = \emptyset$ for $i \neq j$. Consider a grid $\alpha = x_0 < \cdots < x_N < x_{N+1} = \beta,$ i.e. $\alpha_i = x_i$, $\beta_i = x_{i+1}$ for $i \in \{0, \ldots, N\}$. The $\{x_i\}$ are called nodes of the grid. $h_i := x_{i+1} - x_i$ for $i \in \{0, \ldots, N\}$ and $h := \max_i h_i$. $V_h$? Basis?