Today

- ID FE < code
- 2D FE < code

Announcements

- HW5 out this weekend
- Proj. 1 due Wed
Finite Elements in 1D: Discrete Form

\[ \Omega := [\alpha, \beta] \]. Look for \( u \in H_0^1(\Omega) \), so that \( a(u, \varphi) = \langle f, \varphi \rangle \) for all \( \varphi \in H_0^1(\Omega) \). Choose \( V_h = \text{span}\{\varphi_1, \ldots, \varphi_n\} \) and expand \( u_h = \sum_{i=1}^{n} u^i \varphi_i \in V_h \). Find the discrete system.

\[
-u'' = f \\
a(u, \varphi) = \langle f, \varphi \rangle \\
\sum_{i=1}^{n} a(\varphi_i, \varphi_j) = \langle f, \varphi_i \rangle \\
\Rightarrow \text{LauS}
\]
Grids and Hats

Let \( I_i := [\alpha_i, \beta_i] \), so that \( \bar{\Omega} = \bigcup_{i=0}^{N} I_i \) and \( I_i \cap I_j = \emptyset \) for \( i \neq j \). Consider a grid

\[
\alpha = x_0 < \cdots < x_N < x_{N+1} = \beta,
\]
i.e. \( \alpha_i = x_i, \beta_i = x_{i+1} \) for \( i \in \{0, \ldots, N\} \). The \( \{x_i\} \) are called nodes of the grid. \( h_i := x_{i+1} - x_i \) for \( i \in \{0, \ldots, N\} \) and \( h := \max_i h_i \). \( V_h \)? Basis?
Degrees of Freedom and Matrices

Define something more general than basis coefficients to solve for.

\[ y_i : V_n \to \mathbb{R} \quad \text{degrees of freedom} \quad \text{Requirement: } (y_i(v_n) = \delta_i)_{i=1}^N \]

\[ g_0(v_n) = f(x_0) \quad g_1(v_n) = f'(x_0) \quad f(v_n) = f(x) \]

Now express the solve, recalling \( u_h = \sum_{i=1}^N u_i^h \phi_i \).

\[ \sum_{i=1}^N \delta_j^i (\hat{\phi}_j^i) a(\hat{\phi}_j^i, \phi_i) = \langle f, \phi_i \rangle \quad \text{Define a new basis } \hat{\phi}_j^i \text{ so that } y_i(\hat{\phi}_j^i) = \delta_j^i \]

Anything special about the matrix?

\[ a_{i,j} = \int_{\Omega} \phi_i \phi_j \, d\Omega \quad A_{i,j} \neq 0 \text{ only} \]
According to Céa, what’s our main missing piece in error estimation now?

\[
I_n': C^0(\overline{\Omega}) \rightarrow P_n'
\]

\[
\forall i \rightarrow \sum_{i=1}^{N} \delta_i(v) \varphi_i \in P_n'
\]
Interpolation Error \[(\text{ID only})\]

For \(v \in H^2(\Omega)\),

\[
\| v - I_h v \|_{L^2} \leq C h^2 \| D^2 v \|_{L^2} \\
\| D v (v - I_h v) \|_{L^2} \leq C h \| D v \|_{L^2}
\]

If \(v \in H^1(\Omega) \setminus H^2(\Omega)\),

\[
\| v - I_h v \|_{L^2} \leq Ch \| D^2 v \|_{L^2} \\
\lim_{h \to 0} \| D v (v - I_h v) \|_{L^2} \to 0
\]

Is \(I_h^1\) defined for \(v \in H^2\)? for \(v \in H^1 \setminus H^2\)?

\(H^2 \to C^0\) is continuous (dep. on domain, dim.,...) Smooth embedding thm.
Interpolation Error: Towards an Estimate

Provide an \textit{a-priori} estimate.

\[
\| u - u_h \|_{H^1} \leq \frac{c_i}{c_0} \inf_{v \in P_h} \| u - v \|_{H^1} \leq \frac{c_i}{c_0} \| u - I_h^1 u \|_{H^1} \leq C h \| u \|_{H^2}.
\]

What’s the relationship between \( I_h^1 u \) and \( u_h \)?

\textit{None}.
Local-to-Global

Is there a simple way of constructing the polynomial basis?

\[ \text{Local map: } T_i : \mathbb{F} \rightarrow I_i \]
Local-to-Global: Math

Construct a polynomial basis using this approach.

\[ \hat{\varphi}_0 (r) = 1 - r \]

\[ \hat{\varphi}_1 (r) = r \]

\[ \varphi_i (x) = \begin{cases} (\hat{\varphi}_1 \circ T_{i-1}) x & x \in I_{i-1} \\ (\hat{\varphi}_0 \circ T_i) (x) & x \in I_i \end{cases} \]
Demo: Developing FEM in 1D
Possible extension:

\[ P_h^k = \{ v_h \in C(\Omega) : (v_h)_{\mathcal{T}_i} \in P^k \} \]

Higher Order Approximation

Let \( 0 \leq \ell \leq k \). Then for \( v \in H^{\ell+1}(\Omega) \),

\[
\| v - \Pi_h^k v \|_{L^2} + \| D_\omega (v - \Pi_h^k v) \|_{L^2} \leq C h^{\ell+1} \| D_{\omega} v \|_{L^2}.
\]
Define some degrees of freedom (or DoFs) for high-order 1D FEM.