Today
- 207E assembly

Announcements
- HW5 out
- Project 1 due
A Boundary Value Problem

Consider the following elliptic PDE

\[-\nabla \cdot (\kappa(x) \nabla u) = f(x) \quad \text{for } x \in \Omega \subset \mathbb{R}^2,\]

\[u(x) = 0 \quad \text{when } x \in \partial \Omega.\]

Weak form?

\[\forall u \in \mathcal{H}^1(\Omega), \quad v \in \mathcal{H}_0^1(\Omega) : \]

\[\text{\(\int_{\Omega} -\nabla \cdot (\kappa \nabla u) v = \int_{\partial \Omega} \nu \cdot n u\)}\]

\[= \int_{\Omega} v \kappa \nabla u \cdot \nabla u + \int_{\Omega} \kappa \nabla u \cdot \nabla v = \int_{\partial \Omega} \nu u \chi_0\]

\[v \, \kappa \, \nabla \chi_0\]
Weak Form: Bilinear Form and RHS Functional

Hence the problem is to find \( u \in V \), such that

\[
a(u, v) = g(v), \quad \text{for all } v \in V = H_0^1(\Omega)
\]

where...

\[
a(u, v) = \int_\Omega \nabla u(x) \cdot \nabla v(x) \, dx
\]

\[
g(v) = \int_\Omega f(x) v(x) \, dx
\]

Is this symmetric, coercive, and continuous?

- Symmetry: \( \forall u, v \in V \)
- Coercivity: \( 0 < c \leq \forall u \in V \)
- Continuity: \( \forall v \leq C \)
Suppose the domain is a union of triangles $E_m$, with vertices $x_i$. 

$$\Omega = E_1 \cup E_2 \cup \cdots \cup E_{10}$$
Elements and the Bilinear Form

If the domain, $\Omega$, can be written as a disjoint union of elements, $E_k$,

$$\Omega = \bigcup_{m=1}^{M} E_m \text{ with } E_i^\circ \cap E_j^\circ = \emptyset \text{ for } i \neq j,$$

what happens to $a$ and $g$?

$$a(u,v) = \sum_{m=1}^{M} \int_{E_m} K(x) (\nabla u \cdot \nabla v)$$

$$g(v) = \sum_{m=1}^{M} \int_{E_m} f(x) \cdot v(x) \, dx.$$
Basis Functions

Expand

\[ u_N(x) = \sum_{i=1}^{N_p} u_i \varphi_i, \]

and plug into the weak form.

\[ \sum_{j=1}^{N_p} u_j a(\varphi_j, \varphi_i) = g(\varphi_i) \]
Global Lagrange Basis

Approximate solution $u_h$: Piecewise linear on $\Omega$

The Lagrange basis for $S_h$ consists of piecewise linear $\varphi_i$, with...

\[ \varphi_i(x_i) = 1 \quad \varphi_i(x_j) = 0 \quad (j \neq i) \]
Basis Functions Features

Features of the basis?

- \( \varphi_i \in C^0 \), \( \varphi_i \in H^1 \)

- Restricted to each \( \Omega_i \), \( \varphi_i \) are linear.
Local Basis

What basis functions exist on each triangle?
Local Basis Expressions

Write expressions for the nodal linear basis in 2D.

\[ \varphi_1(r, s) = 1 - r - s \]
\[ \varphi_2(r, s) = r \]
\[ \varphi_3(r, s) = s \]
Higher-Order, Higher-Dimensional Simplex Bases

What’s an $n$-simplex?

\[ \mathbf{r} = (r_1, r_2, \ldots, r_n) \]

\[ r_i \geq 0 \quad \exists r_i = 1 \quad (\Rightarrow \text{barycentric}) \]

Give a higher-order polynomial space on the $n$-simplex:

\[ P^N(\Delta) = \text{span} \{ r_i \cdot A : 0 \leq i + j \leq N \} \]

\[ \prod_{i=1}^{n} r_i \cdot A : \sum A_i \leq N \]

Give nodal sets (on the $\Delta$) for $P^N$ and $\dim P^N$ in general.

\[ P^2 \quad P^4 \]

\[ \begin{array}{c}
\text{dim } P^N = \frac{(N+1)(N+2)}{2} \\
\text{dim } P^4 = 15
\end{array} \]
Finding a Nodal/Lagrange Basis in General

Given a nodal set \((\xi_i)_{i=1}^{N_p} \subset \hat{E}\) (where \(\hat{E}\) is the reference element) and a basis \((\varphi_j)_{j=1}^{N_p} : \hat{E} \to \mathbb{R}\), find a Lagrange basis.

\[
V = \begin{bmatrix}
\varphi_i(\xi_i) & \varphi_{N_p}(\xi_i) \\
\vdots & \vdots \\
\varphi_i(\gamma_{N_p}) & \varphi_{N_p}(\gamma_{N_p})
\end{bmatrix}
\]

\[
\ell_i = \sum_j (v^{-T})_{i,j} \varphi_j
\]

In the tensor product case?
Higher-Order, Higher-Dimensional Tensor Product Bases

What’s a tensor product element? \([0,1]^n\)

\[
\begin{array}{ccc}
\text{\square} & \text{\square} & \text{\square} \\
0 \leq r_i \leq 1
\end{array}
\]

Give a higher-order polynomial space on the \(n\)-simplex: \(\text{TP}_\ell\)

\[
Q^N = \{ \prod r_i^d_i : 0 \leq d_i \leq N \}
\]

\(\rightarrow\) Lagrange basis can be found dim-by-dim

Give the nodal sets (on the quad) for \(Q^N\).

\(Q^2:\)

[Diagram of nodal sets for \(Q^2\)]

\(Q^4:\)

[Diagram of nodal sets for \(Q^4\)]
Construct a mapping $T_m : \hat{E} \rightarrow E_m$ that takes the triangular reference element $\hat{E}$ to a global triangle $E_m$.

$$\nabla_m (r, s) = \left( \frac{x_2 - x_1}{x_3 - x_1} \right) r + \left( \frac{x_3 - x_1}{x_2 - x_1} \right) s + x_1 = \left( \frac{x_2 - x_1}{x_2 - x_1}, \frac{x_3 - x_1}{x_2 - x_1} \right) \begin{pmatrix} r \vspace{0.2cm} \end{pmatrix} + x_1$$

What is the Jacobian of $T_m$?

$$\nabla_m (r, s) = \begin{pmatrix} x_2 - x_1 \ vspace{0.2cm} \\ x_3 - x_1 \ vspace{0.2cm} \end{pmatrix}$$
More on Mappings

Is an affine mapping sufficient for a tensor product element?

Affine maps take $\square$ to parallelograms
Affine: components $x, y \in \mathbb{R}^1$

$T_m : \mathbb{R}^2 \rightarrow \mathbb{R}^2 (x(r,s), y(r,s))$

$TP : \text{components } x, y \in \mathbb{Q}$

How might we accomplish curvilinear elements using the same idea?

Choose $T_m \in (\mathbb{R}^n)^n$ "isoparametric" 

"sub"

"super"
Constructing the Global Basis

Construct a basis on the element $E_m$ from the reference basis $(\hat{\varphi}_j)^{N_p}_{j=1} : E_m \to \mathbb{R}$.

\[ \hat{\varphi}_{m,j}(\vec{x}) = \hat{\varphi}_j(T^{-1}(\vec{x})) \]

What’s the gradient of this basis?

\[ \nabla_x \hat{\varphi}_{n,j} = \left( \frac{d}{dx} \hat{\varphi}_j(T^{-1}(\vec{x})) \right)^T = \left( \frac{d\hat{\varphi}_j}{dr} \right)_{z=T^{-1}(\vec{x})} \cdot \vec{j}_{T}^{-1}(\vec{x}) \]

\[ = \vec{j}_T^{-1}(\vec{x}) \cdot \nabla_{\vec{x}} \hat{\varphi}_j(T^{-1}(\vec{x})) \]
Assembling a Linear System

Express the matrix and vector elements in

\[ \sum_{j=1}^{N_p} u_j a(\varphi_j, \varphi_i) = g(\varphi_i) \quad \text{for } i = 1, \ldots, N_p. \]

\[ a(\hat{\varphi}_i, \hat{\varphi}_j) = \sum_{m=1}^{M} \int_{\Sigma} \kappa(x) \partial \hat{\varphi}_i \partial \hat{\varphi}_j \]

\[ g(\hat{\varphi}_i) = \sum_{m=1}^{M} \int_{\Sigma} f(x) \hat{\varphi}_i \]
Integrals on the Reference Element

Evaluate

$$\int_E \kappa(x) \nabla_x \hat{\varphi}_i(x)^T \nabla_x \hat{\varphi}_j(x) \, dx.$$ 

And now the RHS functional.

$$\int_E \kappa(x) \left( \frac{\partial}{\partial x} \hat{\varphi}_i(x) \right)^T \left( \frac{\partial}{\partial x} \hat{\varphi}_j(x) \right) = S_e \kappa(x)$$

$$\rho^1 = \left( \frac{\partial}{\partial x} \hat{\varphi}_i(x) \right)^T \left( \frac{\partial}{\partial x} \hat{\varphi}_j(x) \right) S_e \kappa(x)$$

$$\int E \left[ \frac{\partial}{\partial x} \hat{\varphi}_i(x) \right] \left( \frac{\partial}{\partial x} \hat{\varphi}_j(x) \right) d^2 \hat{x}.$$
Inhomogeneous Dirichlet BCs

Handle an inhomogeneous boundary condition $u(x) = \eta(x)$ on $\partial \Omega$.

Find a function $u^0 \in H^1(\Omega)$ so that $u^0(x) = \eta(x)$ on $\partial \Omega$.

Define: $\hat{u} = u - u^0$

$\hat{u} \in H^0$

Weak form: $u = \hat{u} + u^0$

$$a(u,v) = a(\hat{u},v) + a(u^0,v)$$

$$a(\hat{u},v) = g(v) - a(u^0,v)$$

\text{lifting argument}