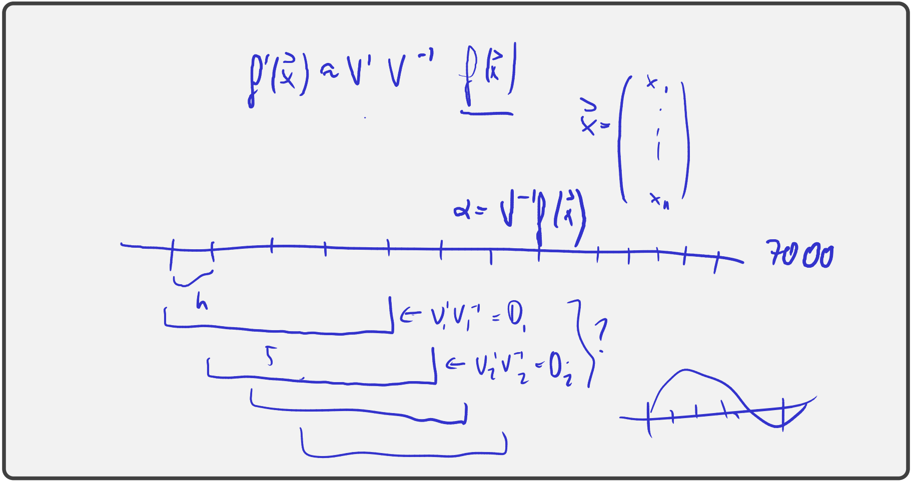
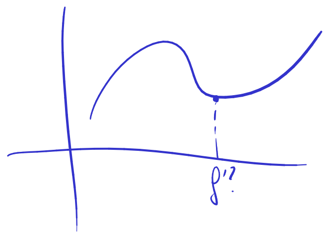
Numerical Differentiation: How?

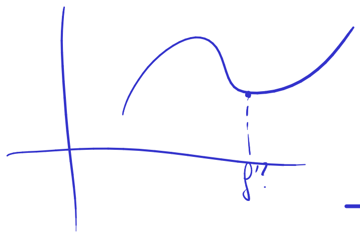
CS555

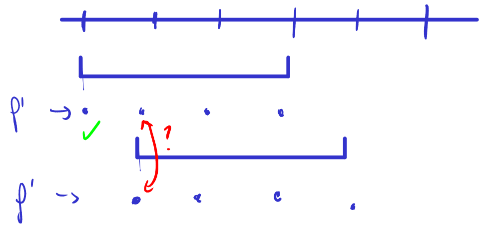
How can we take derivatives numerically?

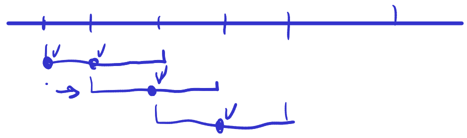


Demo: Taking Derivatives with Vandermonde Matrices [cleared]



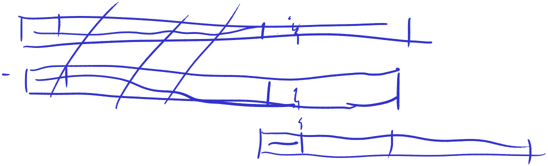






Finite Differences Numerically

Demo: Finite Differences [cleared]
Demo: Finite Differences vs Noise [cleared]
Demo: Floating point vs Finite Differences [cleared]



Taking Derivatives Numerically

Why *shouldn't* you take derivatives numerically?

- ∂ is an unbounded operator

  $\partial : C^1 \to C^0$ $\quad \|x\|$

  $\|\partial\| = \max_{\|f\|_\infty = 1} \|\partial f\|_\infty \qquad \|A\| = \max_{\|x\|=1} \|Ax\|$

  $f_\alpha(x) = e^{i\alpha x} \qquad \|f_\alpha\|_\infty = 1$

  $\|\partial f_\alpha\|_\infty = \|i\alpha e^{i\alpha x}\|_\infty$

- conditioning of ∂

  $\underbrace{\|\partial\|}_{\infty} \underbrace{\|\partial^{-1}\|}_{\nexists}$

- amplifies noise
- loses order of accuracy vs. interp. h^n vs. h^{n-1}

$\Delta u = f \qquad \partial_n u = g$

$u \rightsquigarrow u + c$

Differencing Order of Accuracy Using Taylor

Find the order of accuracy of the finite difference formula
$f'(x) \approx [f(x+h) - f(x-h)]/2h$.

Outline

Truncation Error in Interpolation

If f is n times continuously differentiable on a closed interval I and $p_{n-1}(x)$ is a polynomial of degree at most n that interpolates f at n distinct points $\{x_i\}$ $(i = 1, ..., n)$ in that interval, then for each x in the interval there exists ξ in that interval such that

$$f(x) - p_{n-1}(x) = \frac{f^{(n)}(\xi)}{n!}(x - x_1)(x - x_2)\cdots(x - x_n).$$

Truncation Error in Interpolation: cont'd.

$$Y_x(t) = R(t) - \frac{R(x)}{W(x)} W(t) \quad \text{where} \quad W(t) = \prod_{i=1}^{n} (t - x_i)$$

Error Result: Connection to Chebyshev

What is the connection between the error result and Chebyshev interpolation?

Error Result: Simplified Form

Boil the error result down to a simpler form.

- **Demo:** Interpolation Error [cleared]

Outline

Outline

Introduction

Finite Difference Methods for Time-Dependent Problems
- 1D Advection
- Stability and Convergence
- Von Neumann Stability
- Dispersion and Dissipation
- A Glimpse of Parabolic PDEs

Finite Volume Methods for Hyperbolic Conservation Laws

Finite Element Methods for Elliptic Problems

Discontinuous Galerkin Methods for Hyperbolic Problems

1D Advection Equation and Characteristics

$$u_t + au_x = 0, \quad u(0,x) = g(x) \qquad (x \in \mathbb{R})$$

Solution?

$\hookrightarrow u_t + f(u)_x = 0$

Characteristic curve: Define $x(t)$ so that

$u(x(t),t) = u(x(0),0)$

Suppose x is given by the IVP

$$\begin{cases} \frac{dx}{dt} = f'(u(x(t),t)) \leftarrow \\ x(0) = x_0 \end{cases}$$

$$\frac{du(x(t),t)}{dt} = u_x x'(t) + u_t = u_x f'(u(x(t),t)) + u_t = f(u)_x + u_t = 0$$

Solving Advection with Characteristics

$$u_t + au_x = 0, \quad u(0,x) = g(x) \qquad (x \in \mathbb{R})$$

Find the characteristic curve for advection.

Generalize this to a solution formula.

Does the solution formula admit solutions that aren't obviously allowed by the PDE?