Fourier Transforms of Inverse Toeplitz Operators

Fourier transform $P_h^{-1} Q_h \boldsymbol{y}$?

$$\frac{\hat{q}(\theta)}{\hat{p}(\theta)} \hat{y}(\theta)$$

$$(T\vec{x})_j = \sum_k t_{j-k} x_k \rightsquigarrow \widehat{(T\vec{x})}(\theta) = \hat{t}(\theta)\,\hat{x}(\theta)$$

$$\left(P_h Q_h \vec{x}\right)(\theta) = \hat{p}(\theta)\,\hat{q}(\theta)\,\hat{x}(\theta)$$

$$\left(P_h^{-1} P_h Q_h \vec{x}\right)(\theta) = \frac{1}{\hat{p}(\theta)}\hat{p}(\theta)\,\hat{q}(\theta)\,\hat{x}(\theta) = \hat{q}(\theta)\,\hat{x}(\theta)$$

Bounding the Operator Norm

Bound $\left\|P_h^{-1}Q_h\right\|_2^2$ using Fourier:

$$\|P_h^{-1}Q_h\|_2^2 = \sup_{\vec{x}\neq 0} \frac{\|P_h^{-1}Q_h\vec{x}\|_2^2}{\|x\|_2^2} = \sup_{\vec{x}\neq 0} \frac{\cancel{\frac{h}{2\pi}}\int_{-\pi}^{\pi}\left|\frac{\hat{q}(\theta)}{\hat{p}(\theta)}\hat{x}(\theta)\right|^2 d\theta}{\cancel{\frac{h}{2\pi}}\int_{-\pi}^{\pi}|\hat{x}(\theta)|^2 d\theta}$$

$$\leq \sup_{x\neq 0} \frac{\max_{\varphi\in[-\pi,\pi]}\left|\frac{\hat{q}(\varphi)}{\hat{p}(\varphi)}\right|^2 \cancel{\int_{-\pi}^{\pi}|\hat{x}(\theta)|^2 d\theta}}{\cancel{\int_{-\pi}^{\pi}|\hat{x}(\theta)|^2} d\theta}$$

$$= \max_{\varphi\in[-\pi,\pi]}\left|\frac{\hat{q}(\varphi)}{\hat{p}(\varphi)}\right|^2$$

Analogously $\|P_h^{-1}\|_2^2 \leq \max_{\varphi\in[-\pi,\pi]}\left|\frac{1}{\hat{p}(\varphi)}\right|^2$

Is the upper bound attained?

If $\hat{x}(\theta) = \delta(\theta-\varphi^*)$ where $|\hat{q}(\theta)/\hat{p}(\theta)|$ attains its max, then yes.

$$x_k = \frac{1}{2\pi}e^{i\varphi^* k}$$

von Neumann Stability

Two-level finite difference scheme

$$P_h \boldsymbol{v}_{\ell+1} = Q_h \boldsymbol{v}_\ell + h_t \boldsymbol{b}_\ell,$$

where P_h and Q_h are Toeplitz operators with vectors $\boldsymbol{p}$ and $\boldsymbol{q}$.

Definition (Symbol of a Two-Level Finite Difference Scheme)

Let

$$\hat{\boldsymbol{p}}(\theta) = \sum_k p_k e^{-i\varphi k}, \qquad \hat{\boldsymbol{q}}(\theta) = \sum_k q_k e^{-i\varphi k}.$$

Then the symbol of the two-level FD method is $s(\varphi) = \hat{\boldsymbol{q}}(\varphi)/\hat{\boldsymbol{p}}(\theta)$.

Definition (Von Neumann Stability)

If

$$\max_\varphi |s(\varphi)| \leq 1, \qquad \max_\varphi \left| \frac{1}{\hat{\boldsymbol{p}}(\varphi)} \right| \leq c$$

for some constant $c > 0$, we say the scheme is von Neumann stable.

Comparison with Lax-Richtmyer Stability

Need $\|(P_h^{-1}Q_h)^\ell P_h^{-1}\| \leq c$. $\Leftarrow$

vN stab $\Rightarrow$ Lax-Richtmyer stab $\Rightarrow$ Lax stab.

Why is bounding the symbol the most salient part?

Often, if $\|P_h^{-1}Q_h\|$ has issues so does $\|P_h^{-1}\|$.

Main restriction of von Neumann stability?

- Only works for infinite / periodic domains
- Have BCs? More difficult ($\rightarrow$ Strikwerda)

von Neumann Stability: ETBS (1/2)

$0 \le \lambda \le 1$

ETBS: Let $\lambda = a h_t / h_x$. $u_{k,\ell+1} = \lambda u_{k-1,\ell} + (1-\lambda) u_{k,\ell}$.

$P_h = I \qquad Q_h = \text{tridiag}(\lambda, 1-\lambda, 0)$

Auxiliary result: FT of $r_k = \delta_{k,j}$

$\rightarrow \hat{r}(\varphi) = \sum_k r_k e^{-i\varphi k} = \sum_k \delta_{kj} e^{-i\varphi k} = e^{-i\varphi j}$

$(Q\vec{x})_j = \sum_k x_k q_{j-k} \rightarrow (Q\vec{x})_0 = x_0 q_0 + x_{-1} q_1 + x_1 q_{-1} + \dots$

$\vec{q} = (\dots, 0, 1-\lambda, \lambda, 0 \dots)$

$\hat{p}(\theta) = 1 \qquad \hat{q}(\theta) = \lambda e^{-i\varphi} + (1-\lambda) = 1 - \lambda(1 - e^{-i\varphi})$

$|\hat{s}(\theta)|^2 = \left| \frac{\hat{q}(\theta)}{\hat{p}(\theta)} \right|^2 = (1 - \lambda(1 - e^{-i\varphi}))(1 - \lambda(1 - e^{i\varphi}))$

$= 1 + 2(\lambda - \lambda^2)(\cos\varphi - 1).$

von Neumann Stability: ETBS (2/2)

Found: $|s(\varphi)|^2 = 1 + 2(\lambda - \lambda^2)(\cos\varphi - 1)$.

Take derivative wrt φ

$$\frac{d}{d\varphi}\left(1+2(\lambda-\lambda^2)(\cos\varphi-1)\right) = -2(\lambda-\lambda^2)\sin\varphi \stackrel{!}{=} 0$$

if only if $\varphi \in \mathbb{Z}\pi$

Let $m \in \mathbb{Z}$, $\varphi = m\pi$ $\quad s(m\pi) = 1+2(\lambda-\lambda^2)\left((-1)^m - 1\right)$

For m even, $s(m\pi) = 1$.

For m odd, $s(m\pi) = 1 - 4(\lambda-\lambda^2) = (1-2\lambda)^2$

So $|s(\varphi)|^2 \le 1$ if only if

$$|1-2\lambda| \le \cancel{0}\ 1 \iff 0 \le \lambda \le 1$$

von Neumann Stability: ETCS

Let $\lambda = ah_t/h_x$. Then

$$u_{k,\ell+1} = \frac{\lambda}{2}u_{k-1,\ell} + u_{k,\ell} - \frac{\lambda}{2}u_{k+1,\ell}.$$

$P_n = I \qquad Q_n = \text{tridiag}\left(\frac{\lambda}{2}, 1-\lambda, -\frac{\lambda}{2}\right)$

So $\hat{p}(\theta) = 1$ and

$$\hat{q}(\theta) = \frac{\lambda}{2}e^{-i\varphi} + 1 - \frac{\lambda}{2}e^{-i\varphi(-1)} = 1 - \lambda \sin(\varphi) i$$

$$\max_{\varphi \in [-\pi,\pi]} |s(\varphi)|^2 = \left|\frac{\hat{q}(\varphi)}{\hat{p}(\varphi)}\right|^2 = 1 + \lambda^2 \sin^2(\varphi) \geq 1$$

ETCs is unstable.

von Neumann Stability: Crank-Nicolson

Let $\lambda = ah_t/(4h_x)$

$$-\lambda u_{k-1,\ell+1} + u_{k,\ell+1} + \lambda u_{k+1,\ell+1} = \lambda u_{k-1,\ell} + u_{k,\ell} - \lambda u_{k+1,\ell}.$$

$P_h = \text{tridiag}(-\lambda, 1, \lambda) \qquad Q_h = \text{tridiag}(\lambda, 1, -\lambda)$

$\hat{p}(\varphi) = -\lambda e^{-i\varphi} + 1 + \lambda e^{i\varphi} = 1 + 2\lambda i \sin(\varphi)$

$\hat{q}(\varphi) = \lambda e^{-i\varphi} + 1 - \lambda e^{i\varphi} = 1 - 2\lambda i \sin(\varphi)$

$$|s(\varphi)|^2 = \frac{1 + 4\sin^2(\varphi)}{1 + 4\sin^2(\varphi)} = 1$$

CN is unconditionally stable.

Outline

Studying Solutions of the PDE

$$\frac{d}{dx}\left(e^{\lambda x}\right) = \lambda\left(e^{\lambda}\right)$$

Saw numerically: interesting dispersion/dissipation behavior.
Want: theoretical understanding.

Consider *linear*, *continuous* (not yet discrete) differential operators

$$\begin{aligned} L_1 u &= u_t + a u_x, \\ L_2 u &= u_t - D u_{xx} + a u_x \qquad (D>0) \\ L_3 u &= u_t + a u_x - \mu u_{xxx}. \end{aligned}$$

What could we use as a prototype solution?

A Prototype Solution of the PDE

Observation: all these operators are diagonalized by complex exponentials.
Come up with a 'prototype complex exponential solution'.

$$e^{i(kx-\omega t)}$$

What type of function is this?

Wave-like Solutions of the PDE

$$z(x,t) = z_0 e^{i(kx-\omega t)}$$

Observations in connection with L?

What is the dispersion relation?