Entropy? condition:

$$f'(u^-) > s > f'(u^+) \iff \text{"good shock"}$$

convex flux: $\quad f'(u^-) > f'(u^+)$

Vanishing viscosity: $\quad u \rightsquigarrow u^\varepsilon$

$$(u_\varepsilon)_t + f(u_\varepsilon)_x = \varepsilon u_{xx}$$

Entropy-Flux Pairs

What are features of (physical) entropy?

Definition (Entropy/Entropy Flux)

An entropy $\eta(u)$ and an entropy flux $\psi(u)$ are functions so that η is convex and

$$\eta(u)_t + \psi(u)_x = 0$$

for smooth solutions of the conservation law.

Finding Entropy-Flux Pairs

$\eta(u)_t + \psi(u)_x = 0$. Find conditions on η and ψ.

$$\psi'(u) = \eta'(u) f'(u)$$

Come up with an entropy-flux pair for Burgers.

$f(u) = \frac{u^2}{2}$ $\quad \eta(u) = u^2$ $\quad \psi'(u) = 2u \cdot u$

(one choice) $\psi(u) = 2u^3/3$

Back to Vanishing Viscosity (1/2)

$$u_t + f(u)_x = \varepsilon u_{xx}$$

What's the evolution equation for the entropy?

$$\eta'(u)\, u_t + \eta'(u) f'(u)\, u_x = \varepsilon \eta'(u)\, u_{xx}$$

$$\Leftrightarrow \quad \eta'(u)_t + \Psi(u)_x = \varepsilon\left(\eta'(u) u_x\right)_x$$

$$- \varepsilon \eta''(u) u_x^2$$

Back to Vanishing Viscosity (2/2)

$$\eta(u)_t + \psi(u)_x = \varepsilon(\eta'(u)u_x)_x - \varepsilon\eta''(u)u_x^2.$$

Integrate this over $[x_1, x_2] \times [t_1, t_2]$.

?

$$\int_{t_1}^{t_2}\int_{x_1}^{x_2} \eta(u)_t + \psi(u)_x \, dx\, dt$$

$$= \int_{t_1}^{t_2} \Big[\eta'(u) u_x \Big]_{x_1}^{x_2} dt$$

$$- \varepsilon \int_{t_1}^{t_2} \int_{x_1}^{x_2} \underbrace{\eta''(u) u_x^2}_{\geq 0} dx$$

$$\eta(u)_t + \psi(u)_x \leq 0$$

Entropy Solution

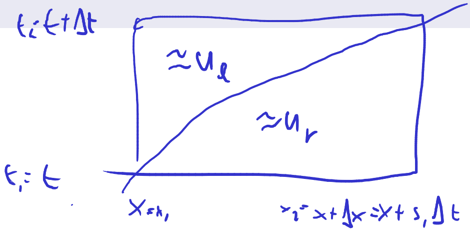
Definition (Entropy solution)

The function $u(x,t)$ is the entropy solution of the conservation law if for all convex entropy functions and corresponding entropy fluxes, the inequality

$$\eta(u)_t + \psi(u)_x \leq 0$$

is satisfied in the weak sense.

$$\Delta x = \underbrace{\frac{[u^2/2]}{[u]}}_{s_1} \Delta t$$



Entropy Solution vs Entropy Condition

Relate entropy solutions $\eta(u)_t + \psi(u)_x \leq 0$ back to the entropy condition.

~~Consider~~ Burgers. $s_1 = [u^2/2]/[u]$

$s_2 = [2u^3/3]/[u^2] \longleftarrow$ "R-H applied to entropy conservation"

$s_2 - s_1 = [u]^2 / [6(u_\ell + u_r)]$.

$$0 \geq \left[\int_{x_1}^{x_2} u^2 \right]_{t_1}^{t_1+\Delta t} + \left[\int_{t_1}^{t_1+\Delta t} 2u^3/3\right]_{x_1}^{x_1+\Delta x}$$

$$= s_1 \Delta t (u_\ell^2 - u_r^2) + \tfrac{2}{3}\Delta t (u_r^3 - u_\ell^3) + O(\Delta t^2)$$

$$= \Delta t (u_\ell^2 - u_r^2)(s_1 - s_2) + O(\Delta t^2)$$

$$= \Delta t (u_\ell^2 - u_r^2)\left[-\frac{1}{6}\frac{[u]^2}{u_\ell + u_r}\right] + O(\Delta t^2)$$

$$= -\tfrac{1}{6}(u_\ell - u_r)^3 \Delta t + O(\Delta t^2) \rightsquigarrow u_\ell \geq u_r$$

Conservation of Entropy?

What can you say about conservation of entropy in time?