Back to Advection

$f^*(u^-, u^+)$

Consider only $f(u) = au$ for now. Riemann solver inspiration from FD?

const. coeff.

$u_t + a u_x = 0$

var. coeff.

$u_t + a(x) u_x = 0$

$$0 = \frac{u_{j,\ell+1} - u_{j,\ell}}{} + \frac{f^*(u_{j,\ell}, u_{j+1,\ell}) - f^* \cdots}{h_x}$$

$$f^*(u^-, u^+) = \begin{cases} a u^- & a \geq 0 \\ a u^+ & a < 0 \end{cases} = \underline{\frac{a u^- + a u^+}{2} - \frac{|a|}{2}(u^+ - u^-)}$$

var. coeff.: which a?

Side Note: First Order Upwind, Rewritten

$$\frac{u_{j,\ell+1} - u_{j,\ell}}{h_t} + \frac{f^*(u_{j,\ell}, u_{j+1,\ell}) - f^*(u_{j-1,\ell}, u_{j,\ell})}{h_x}$$

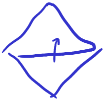
with

$$f^*(u^-, u^+) = \frac{au^- + au^+}{2} - \underline{\frac{|a|}{2}(u^+ - u^-)}.$$

$$\underbrace{\frac{u_{j,\ell+1} - u_{j,\ell}}{h_t} + a\frac{u_{j+1,\ell} - u_{j-1,\ell}}{2h_x}}_{\text{FTCS}} = \frac{ah_x}{2} \underbrace{\frac{u_{j+1,\ell} - 2u_{j,\ell} + u_{j-1,\ell}}{h_x^2}}_{= u_{xx} + O(h_x^2)}$$

Lax-Friedrichs "width" $\to [u]$ / $[f] = [u]\cdot[x]/[t]$

Generalize linear upwind flux for a nonlinear conservation law:



$$f^*(u^-, u^+) = \frac{au^- + au^+}{2} - \frac{|a|}{2}(u^+ - u^-).$$

$$f^*(u^-, u^+) = \underbrace{\underbrace{\frac{f(u^-) + f(u^+)}{2}}_{\text{"central flux"}}}_{[u]\cdot[x]/[t]} - \frac{\alpha}{2}\underbrace{(u^+ - u^-)}_{[u]}$$

$$u_t + f(u)_x = 0$$

$$\frac{[u]}{[t]} + \left[\frac{[u]\cdot[x]}{[t]}\right]\cdot\frac{1}{[x]}$$

$$[\alpha] = \frac{[x]}{[t]}$$

Attempt 1: $\cancel{\alpha = f'\left(\frac{u_- + u_+}{2}\right)}$

can conv. to sol that do not obey ent. cond.

Demo: Finite Volume Burgers [cleared] (Part I)

Attempt 2: $\alpha = \max(|f'(u^-)|, |f'(u^+)|)$

"local Lax-Friedrichs" flux / Rusanov

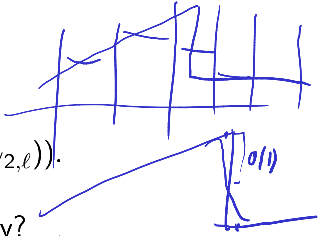
"global L-F": take max over whole solution

(global : more dissipative)

Outline

Improving Accuracy

Consider our existing discrete FV formulation:



$$\bar{u}_{j,\ell+1} = \bar{u}_{j,\ell} - \frac{h_t}{h_x}\left(f(u_{j+1/2,\ell}) - f(u_{j-1/2,\ell})\right).$$

What obstacles exist to increasing the order of accuracy?

- Time discr.
- Spatial discr.
- Non-smoothness

↳ ℓ^∞: conv. $O(1)$

ℓ^2: conv. $O(h)$

What order of accuracy can we expect?

Near a discontinuity: won't be able to do better than first order

Goal: more accuracy <u>away</u> from the shock

Improving the Order of Accuracy

Improve temporal accuracy.

$$\frac{d\bar{u}}{dt}(t) + \frac{f^*(u^-_{j+\frac{1}{2}}, u^+_{j+\frac{1}{2}}) - f^*(u^-_{j-\frac{1}{2}}, u^+_{j-\frac{1}{2}})}{h_x} = 0$$

↳ Idea: use higher-order method on that ODE

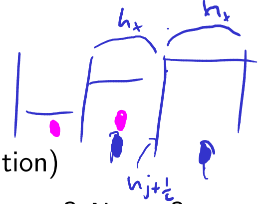
What's the obstacle to higher spatial accuracy?

Letting $u^-_{j+\frac{1}{2}} = \bar{u}_j = u^+_{j-\frac{1}{2}}$.

How can we improve the accuracy of that approximation?

Involve more cells in the reconstruction!

Increasing Spatial Accuracy



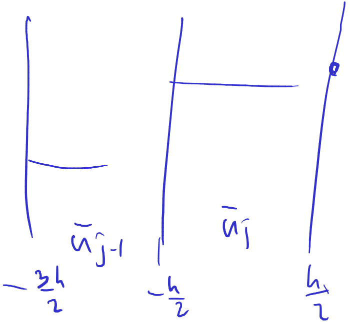
Temporary Assumptions:

- $f'(u) \geq 0$
- $f^*_{j+1/2} = f(\bar{u}_j)$ (e.g. Godunov in this situation)

Reconstruct $u_{j+1/2}$ using $\{\bar{u}_{j-1}, \bar{u}_j, \bar{u}_{j+1}\}$. Accuracy? Names?

- $u^{(1)}_{j+\frac{1}{2}} = \frac{1}{2}(\bar{u}_j + \bar{u}_{j+1})$ 2nd order central
- $u^{(2)}_{j+\frac{1}{2}} = \frac{3}{2}\bar{u}_j - \frac{1}{2}\bar{u}_{j-1}$ 2nd order upwind

Compute fluxes, use increments over cell average:



$$u \approx ax + b$$

$$\frac{1}{h_x}\int_{-\frac{3h}{2}}^{-\frac{h}{2}} ax+b = \bar{u}_{j-1}$$

$$\frac{1}{h_x}\int_{-\frac{h}{2}}^{\frac{h}{2}} ax+b = \bar{u}_j$$

$$u\left(\frac{h}{2}\right) =$$

Lax-Wendroff

For $u_t + au_x$, from finite difference:

$$f^*(u^-, u^+) = \frac{au^- + au^+}{2} - \frac{a^2}{2} \cdot \frac{\Delta t}{\Delta x}(u^+ - u^-).$$

Taylor in time: $u_{\ell+1} = u_\ell + \partial_t u_\ell \cdot h_t + \partial_t^2 u_\ell \cdot h_t/2 + O(h_t^3)$.

$$\frac{u_{j,\ell+1} - u_{j,\ell}}{h_t} + \frac{f(u_{j+1,\ell}) - f(u_{j-1,\ell})}{2h_x}$$
$$= \frac{h_t}{2h_x}\left[f'(u_{j+1/2,\ell})\frac{f(u_{j+1,\ell}) - f(u_{j,\ell})}{h_x} - f'(u_{j-1/2,\ell})\frac{f(u_{j,\ell}) - f(u_{j-1,\ell})}{h_x}\right]$$

As a Riemann solver:

$$f^*(u^-, u^+) = \frac{f(u^-) + f(u^+)}{2} - \frac{h_t}{h}[f'(u^\circ)(f(u^+) - f(u^-))].$$