$$\rightsquigarrow u_t + f(u)_x = 0$$

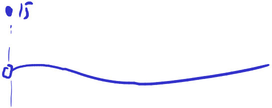
$$u(x,0) = \begin{cases} u_l & x<0 \\ u_r & x>0 \end{cases}$$

$\in \mathbb{R}^n$

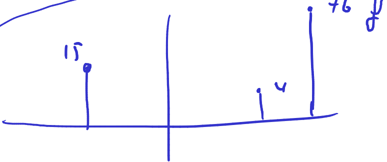
$\ell^2 \qquad \|\vec{x}\|_2 = \sqrt{\sum_i |x_i|^2}$

$L^2 \qquad \|f\| = \sqrt{\int_{-\pi}^{\pi} |f(\xi)|^2 \, d\xi}$

$f: [-\pi,\pi] \to \mathbb{R}$



$\|f\|_2 = 0 \Rightarrow f \equiv 0$

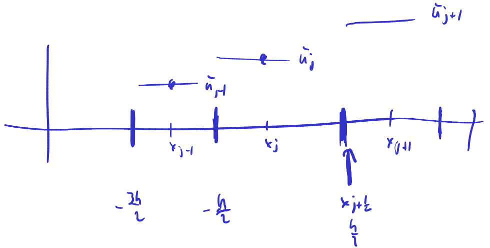


$\forall x \in V: \quad \|x\| \geq 0$

$\forall \alpha \in \mathbb{R}, x \in V: \quad \|\alpha x\| = |\alpha| \, \|x\|$

$\forall x, y \in V: \quad \|x+y\| \leq \|x\| + \|y\|$

$\forall x \in V: \quad \|x\| = 0 \Leftrightarrow x = 0$



$\tilde{u}_{j+1}$
$\tilde{u}_j$
$\tilde{u}_{j-1}$
x_{j-1}
x_j
x_{j+1}
$-\frac{3h}{2}$
$-\frac{h}{2}$
$x_{j+\frac{1}{2}}$
$\frac{h}{2}$

Increasing Spatial Accuracy

Temporary Assumptions:

- $f'(u) \geq 0$ $\to f^*_{j+\frac{1}{2}}(u^-, u^+) = u^-$
- $\cancel{f^*_{j+1/2} = f(\bar{u}_j)}$ (e.g. Godunov in this situation)

Reconstruct $u_{j+1/2}$ using $\{\bar{u}_{j-1}, \bar{u}_j, \bar{u}_{j+1}\}$. Accuracy? Names?

$$u^{-(1)}_{j+\frac{1}{2}} = (\bar{u}_j + \bar{u}_{j+1})/2 \qquad (\text{"central"})$$

$$u^{-(2)}_{j+\frac{1}{2}} = \frac{3}{2}\bar{u}_j - \frac{\bar{u}_{j-1}}{2} \qquad (\text{"upwind"})$$

Compute fluxes, use increments over cell average:

$$f^{*(1)}_{j+\frac{1}{2}} = f\Big(\bar{u}_j + \underbrace{\frac{\bar{u}_{j+1} - \bar{u}_j}{2}}_{\tilde{u}_j^{(1)}}\Big) \qquad f^{(2)}_{j+\frac{1}{2}} = f\Big(\bar{u}_j + \underbrace{\frac{\bar{u}_j - \bar{u}_{j-1}}{2}}_{\tilde{u}_j^{(2)}}\Big)$$

Lax-Wendroff

For $u_t + au_x$, from finite difference:

$$f^*(u^-, u^+) = \frac{au^- + au^+}{2} - \frac{a^2}{2} \cdot \frac{\Delta t}{\Delta x}(u^+ - u^-).$$

Taylor in time: $u_{\ell+1} = u_\ell + \partial_t u_\ell \cdot h_t + \partial_t^2 u_\ell \cdot h_t/2 + O(h_t^3)$.

$u_t = -f(u)_x$ "Kowalewskaya trick"

$u_{tt} = (u_t)_t = (-f(u)_x)_t = (-f(u)_t)_x = -(f'(u)\, u_t)_x = -(f'(u)\, f(u)_x)_x$

$$\frac{u_{j,\ell+1} - u_{j,\ell}}{h_t} + \frac{f(u_{j+1,\ell}) - f(u_{j-1,\ell})}{2h_x}$$

← central for everybody

$$= \frac{h_t}{2h_x}\left[f'(u_{j+1/2,\ell})\frac{f(u_{j+1,\ell}) - f(u_{j,\ell})}{h_x} - f'(u_{j-1/2,\ell})\frac{f(u_{j,\ell}) - f(u_{j-1,\ell})}{h_x}\right]$$

As a Riemann solver:

$$f^*(u^-, u^+) = \frac{f(u^-) + f(u^+)}{2} - \frac{h_t}{h}[f'(u^\circ)(f(u^+) - f(u^-))].$$

Monotone Schemes

Definition (Monotone Scheme)

A scheme

$$\begin{aligned} u_{j,\ell+1} &= u_{j,\ell} - \lambda(f^*(u_{j-p}, \ldots, u_{j+q}) - f^*(u_{j-p-1}, \ldots, u_{j+q-1})) \\ &=: G(u_{j-p-1}, \ldots, u_{j+q}) \end{aligned}$$

is called a montone scheme if G is a monotonically nondecreasing function $G(\uparrow, \uparrow, \ldots, \uparrow)$ of each argument.

Monotonicity for Three-Point Schemes

Three-Point Scheme:

$\lambda \geq 0$

$$G(u_{j-1}, u_j, u_{j+1}) = u_j - \lambda[f^*(u_j, u_{j+1}) - f^*(u_{j-1}, u_j)].$$

When is this monotone?

If $f^*(\uparrow, \downarrow)$, then clearly $G(\uparrow, \quad, \uparrow)$

$\partial_2 f^*(x, \cdot) \leq 0$

$\partial_3 G(u_{j-1}, u_j, u_{j+1}) = -\lambda \underbrace{\partial_2 f^*(u_j, u_{j+1})}_{\leq 0} \geq 0$

$\partial_1 G(\ldots) \geq 0$

$$\frac{\partial_2 G}{\partial u_j} = 1 - \lambda(\partial_1 f^* - \partial_2 f^*) \geq 0$$

$$\Leftrightarrow \quad \lambda(\partial_1 f^* - \partial_2 f^*) \leq 1 \quad \Leftarrow \text{CFL}$$

Lax-Friedrichs is Monotone

$\overline{\text{LLF}}$ / Rusanov: $\alpha = \max(f'(u^-), f'(u^+))$

$$f^*(u^-, u^+) = \frac{f(u^-) + f(u^+)}{2} - \frac{\alpha}{2}(u^+ - u^-).$$

Show: This is monotone.

$$\partial_1 f^* = \frac{1}{2}\left(f'(u^-) + \alpha\right) \geq 0$$

$$\partial_2 f^* = \frac{1}{2}\left(f'(u^+) - \alpha\right) \leq 0$$

Monotone Schemes: Properties

Theorem (Good properties of monotone schemes)

- *Local maximum principle:*

$$\min_{i \in \text{stencil around } j} u_i \leq G(u)_j \leq \max_{i \in \text{stencil around } j} u_i.$$

- *L^1-contraction:*

$$\|G(u) - G(v)\|_{L^1} \leq \|u - v\|_{L^1}.$$

- *TVD:*

$$TV(G(u)) \leq TV(u).$$

- *Solutions to monotone schemes satisfy all entropy conditions.*

Godunov's Theorem

Theorem (Godunov)

Monotone schemes are at most first-order accurate.

What now?