$W_{j,Q_{\dagger}} = G(M_{j,P,Q_{\dagger}} - M_{j,P,Q})$ 'monotone' ? ? (uf):,

Monotone Schemes: Properties

Theorem (Good properties of monotone schemes)

Local maximum principle:

$$\min_{i \in stencil \text{ around } j} u_i \leq G(u)_j \leq \max_{i \in stencil \text{ around } j} u_i$$

► L¹-contraction:

$$\|G(u) - G(v)\|_{L^1} \le \|u - v\|_{L^1}$$
.

► TVD:

$$TV(G(u)) \leq TV(u).$$

Solutions to monotone schemes satisfy all entropy conditions.

Godunov's Theorem

Theorem (Godunov)

Monotone schemes are at most first-order accurate.

What now?

Linear Schemes

Definition (Linear Schemes)

A scheme is called a linear scheme if it is linear when applied to a linear PDE:

$$u_t + au_x = 0,$$

where *a* is a constant.

Write the general case of a linear scheme for $u_t + u_x = 0$:

$$W_{j,R+1} = \sum_{k=-k}^{K} c_{k}(\lambda) W_{j-k,R} - G(\ldots)$$

monotone (c) all $c_{k}(\lambda) > 0$
 C_{j} linear + monotone (c) " positive"

Linear + TVD = ?

Theorem (TVD for linear Schemes)

For linear schemes, $TVD \Rightarrow$ monotone.

What does that mean?

Now what?

Try for nonlinear schemes ...?
Neved to prove TVD ...
$$TV(\tilde{u}^{0}) = C[u_{j+1,\ell} - u_{j,\ell}]$$

Harten's Lemma

Theorem (Harten's Lemma)

If a scheme can be written as

$$ar{u}_{j,\ell+1} = ar{u}_{j,\ell} + \lambda (C_{j+1/2}\Delta_+ ar{u}_j - D_{j-1/2}\Delta_- ar{u}_j)$$

with $C_{j+1/2} \ge 0$, $D_{j+1/2} \ge 0$, $1 - \lambda(C_{j+1/2} + D_{j+1/2}) \ge 0$ and $\lambda = h_t/h_x$, then it is TVD.

As a matter of notation, we have

$$\Delta_+ u_j = u_{j+1} - u_j,$$

$$\Delta_- u_j = u_j - u_{j-1}.$$

We have omitted the time subscript for the time level ℓ .

Harten's Lemma: Proof

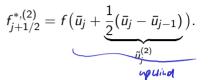
$$\begin{split} & \int_{+} u_{j_{1}e_{1}} = \int_{+} u_{j_{1}e_{1}} + \lambda \int_{+} (C_{j_{1}+\frac{1}{2}} \Delta_{+} u_{j_{1}} - D_{j_{1}+\frac{1}{2}} \Delta_{-} u_{j}) \\ &= \int_{+} u_{j_{1}} + \lambda \Big(C_{j_{1}+\frac{1}{2}} \Delta_{+} u_{j_{1}} - D_{j_{1}+\frac{1}{2}} \Delta_{-} u_{j}) \\ &- C_{j_{1}+\frac{1}{2}} \int_{+} u_{j_{1}} + D_{j_{1}+\frac{1}{2}} \int_{-} u_{j} \Big) \\ &+ (1 - \lambda \Big(C_{j_{1}+\frac{1}{2}} + D_{j_{1}+\frac{1}{2}} \Big) \Big) \Delta_{+} u_{j} \\ &+ \lambda C_{j_{1}+\frac{3}{2}} \int_{+} u_{j_{1}+1} + \lambda D_{j_{1}+\frac{1}{2}} \int_{-} u_{j} \Big) \\ & \int_{+} u_{j_{1}(1)} \Big| \leq \left(1 - \lambda \Big(C_{j_{1}+\frac{1}{2}} + D_{j_{1}+\frac{1}{2}} \Big) \Big) \Big| \Delta_{+} u_{j} \Big) \\ &+ \lambda C_{j_{1}+\frac{3}{2}} \Big[\Delta_{+} u_{j_{1}+1} + \lambda D_{j_{1}+\frac{1}{2}} \Big] \int_{-} u_{j} \Big| \\ &+ \lambda C_{j_{1}+\frac{3}{2}} \Big[\Delta_{+} u_{j_{1}+1} + \lambda D_{j_{1}+\frac{1}{2}} \Big] \int_{-} u_{j} \Big| \\ &+ \lambda C_{j_{1}+\frac{3}{2}} \Big[\Delta_{+} u_{j_{1}+1} + \lambda D_{j_{1}+\frac{1}{2}} \Big] \int_{-} u_{j} \Big| \\ &+ \lambda C_{j_{1}+\frac{3}{2}} \Big[\Delta_{+} u_{j_{1}+1} + \lambda D_{j_{1}+\frac{1}{2}} \Big] \Big| \Delta_{+} u_{j} \Big| \\ &= \int_{-}^{+} \nabla \Big(u_{j}^{e_{1}} \Big) \\ &= \int_{-}^{+} \nabla \Big(u_{j}^{e_{1}} \Big) \end{aligned}$$

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Minmod Scheme

Still assume $f'(u) \ge 0$.

$$f_{j+1/2}^{*,(1)} = f(\bar{u}_j + \underbrace{\frac{1}{2}(\bar{u}_{j+1} - \bar{u}_j)}_{\prod_{j=1}^{j}}),$$



Design a 'safe' thing to use for \tilde{u} :

$\mathsf{Minmod} \text{ is } \mathsf{TVD}$

Show that Minmod is TVD:

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