$$\vec{u}_{\ell+1} = \vec{u}_\ell + h L(\vec{u}) \quad \leftarrow \text{FE}$$

$$HL \Rightarrow \quad TV(\vec{u}_{\ell+1}) < TV(\vec{u}_\ell) \quad \Leftarrow$$

SSP methods are conv. comb. of FE steps

$$\begin{array}{c|ccc|l} & \partial_x & \partial_y & \partial_z & \\ & \downarrow & \downarrow & \downarrow & \\ f\begin{pmatrix} E \\ H \end{pmatrix} = & & & & \leftarrow E_1 \\ & & & & \leftarrow E_2 \\ & & & & \leftarrow H_1 \end{array}$$

$$f : \mathbb{R}^6 \to \mathbb{R}^{6\times 3}$$

$$\vec{u}_t + \nabla \cdot f(\vec{u}) = 0$$

Euler's eqns of gas dyn.
Wave eqn.
Maxwell's eqns.

$$\partial_t \vec{E} + c \cdot \nabla \times H = 0$$

$$\partial_t \vec{H} + c \cdot \nabla \times E = 0$$

Systems of Conservation Laws

Linear system of hyperbolic conservation laws, $A \in \mathbb{R}^{m \times m}$:

$$\begin{aligned} \boldsymbol{u}_t + A\boldsymbol{u}_x &= 0, \\ \boldsymbol{u}(x,0) &= \boldsymbol{u}_0(x). \end{aligned}$$

$AR = R\Lambda$

Assumptions on A?

$\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_m)$

Hyperbolic $\Leftrightarrow$ A is diagonalizable w/ real eigenvalues.

$A\vec{r}_p = \lambda_p \vec{r}_p$

Called strictly hyperbolic if eigenvalues are all distinct.

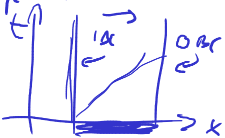
$\vec{v} = R^{-1}u$

$R^{-1}\vec{u}_t + R^{-1}ARR^{-1}\vec{u}_x = 0$

$\vec{v}_t + \Lambda\vec{v}_x = \vec{0}$

Linear System Solution

$q_t + a q_x = 0$



$$\boldsymbol{v} = R^{-1}\boldsymbol{u}, \qquad \boldsymbol{v}_t + \Lambda \boldsymbol{v}_x = 0.$$

Write down the solution.

$$u(x,t) = \sum_p \vec{r}_p \underline{v_p(x - \lambda_p t, 0)}$$

$$\vec{v}(x,0) = R^{-1}\vec{u}(x,0)$$

What is the impact on boundary conditions? E.g. $(\lambda_p) = \underline{(-c, 0, c)}$ for a BC at $x = 0$ for $[0, 1]$?

Can only impose BCs on characteristics "into the domain". w/ state may vary

in nonlinear problems: HBCs

Characteristics for Systems (1/2)

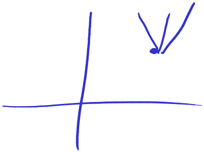
Consider system $\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x = 0$. Write in quasilinear form:

$$\vec{u}_t + J_f(u)\,\vec{u}_x = 0$$

When hyperbolic?

if $J_f(\vec{u})$ is diag w/ real eigvals.
strictly hyperbolic if eigenvalues are distinct.

Characteristics for Systems (2/2)



What about characteristics/shock speeds?

- By considering char. decomp., can still define characteristics. m characteristics through each pt.
- No longer an ODE for char. curves.

Are values of $\boldsymbol{u}$ still constant along characteristics?

No, only char. coefficients are "locally constant".

Shocks and Riemann Problems for Systems

$$
\begin{aligned}
\boldsymbol{u}_t + A\boldsymbol{u}_x &= 0, \\
\boldsymbol{u}(x,0) &= \begin{cases} \boldsymbol{u}_l & x < 0, \\ \boldsymbol{u}_r & x > 0. \end{cases}
\end{aligned}
$$

Solution? (Assume strict hyperbolicity with $\lambda_1 < \lambda_2 < \cdots < \lambda_m$.)

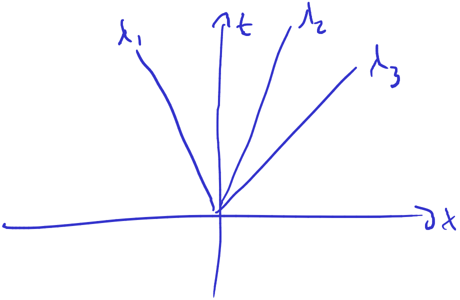
$$\vec{u}_l = \sum_p \alpha_p \vec{r}_p \qquad \vec{u}_r = \sum_p \beta_p \vec{r}_p \quad \Rightarrow v_p(x,0) = \begin{cases} \alpha_p & x < 0 \\ \beta_p & x > 0 \end{cases}$$

Let $P(x,t)$ be the max. value of p so that $x - \lambda_p t > 0$

$$\vec{u}(x,t) = \sum_{p=1}^{P(x,t)} \beta_p \vec{r}_p + \sum_{p=P(x,t)+1}^{m} \alpha_p \vec{r}_p$$

Shock Fans (1/2)

What does the solution look like?



Jump across the characteristic associated with λ_p?

$$[\vec{u}] = (\beta_p - \alpha_p)\,\vec{r}_p$$

Shock Fans (2/2)

Do those jumps satisfy Rankine-Hugoniot?

$$[\vec{f}] = A[\vec{u}] = (\beta_p - \alpha_p) A \vec{r}_p = (\beta_p - \alpha_p)\lambda \vec{r}_p$$
$$= \lambda_p [\vec{u}]$$

How can we find intermediate values of $\boldsymbol{u}$?

$$[u] = u_r - u_\ell = (\beta_1 - \alpha_1)\vec{r}_1 + \dots + (\beta_m - \alpha_m)\vec{r}_m$$

Can use R-H as constraint on ↑ to find intermed. states.

Two Dimensions

$u_t + f(u)_x + g(u)_y = 0$. Finite volume methods generalize in principle:

$$\frac{d\bar{u}_{ij}}{dt} + \frac{1}{h^2}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} f(u(x_{i+\frac{1}{2}}, y, t)) - f(u(x_{i-\frac{1}{2}}, y, t))\,dy$$

$$+ \frac{1}{h^2}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} (\text{analogous})\,dx$$

However:

- 2D: TVD $\Rightarrow$ first order [Goodman/LeVeque '85]
- Reconstruction stencils have a tendency to get dense $\Rightarrow$ expensive

Outline