Weak Derivatives

Define the space L^1_{loc} of locally integrable functions.

$$L^1_{loc} = \{ u : (u : \Omega \to \mathbb{R}) \text{ measurable}$$
$$\int |u(x)\varphi(x)|\, dx < \infty \text{ for all } \varphi \in C_0^\infty(\Omega) \}$$

Definition (Weak Derivative)

$v \in L^1_{\text{loc}}(\Omega)$ is the weak partial derivative of $u \in L^1_{\text{loc}}(\Omega)$ of multi-index order $\boldsymbol{k}$ if

$$D^k_w \quad \int \underset{\text{"}D^k u\text{"}}{v}\, \varphi\, dx = (-1)^{|k|} \int u\, D^k \varphi\, dx \quad \text{for all } \varphi \in C_0^\infty(\Omega)$$

In this case $D^{\boldsymbol{k}} u := v$

Weak Derivatives: Examples (1/2)

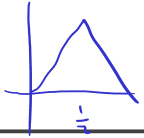
Consider all these on the interval $[-1, 1]$.

$4x - 4x^2$

$$f_1(x) = 4(1-x)x$$

$4 - 8x$

$$f_2(x) = \begin{cases} 2x & x \leq 1/2, \\ 2 - 2x & x > 1/2. \end{cases}$$



$$D_w f_2 = \begin{cases} 2 & x \leq \frac{1}{2} \\ -2 & x > \frac{1}{2} \end{cases}$$

Weak Derivatives: Examples (2/2)

$$f_3(x) = \sqrt{\frac{1}{2}} - \sqrt{|x - 1/2|}$$

$$D_w f_3 = \begin{cases} \frac{1}{2\sqrt{1/2 - x}} & x < \frac{1}{2} \\ -\frac{1}{2\sqrt{x - 1/2}} & x > \frac{1}{2} \end{cases}$$

Sobolev Spaces

Let $\Omega \subset \mathbb{R}^n$, $k \in \mathbb{N}_0$ and $1 \leq p < \infty$.

Definition ((k, p)-Sobolev Norm/Space)

Sobolev norm $\quad \|u\|_{k,p} = \sqrt[p]{\sum_{|\vec{\alpha}| \leq k} \|D_w^{\alpha} u\|_p^p}$

Sobolev seminorm $\quad |u|_{k,p} = \sqrt[p]{\sum_{|\vec{\alpha}| = k} \|D_w^{\alpha} u\|_p^p}$

$W^{k,p} = \{u : \Omega \to \mathbb{R},\ \|u\|_{k,p} < \infty\}$

More Sobolev Spaces

Sam

$W^{0,2}$?

L^2

$W^{s,2}$?

H^s → Hilbert spaces $(u,v)_{H^s} := \sum_{|\alpha| \le s} (D^\alpha_w u, D^\alpha_w v)_{L^2}$

$H^1_0(\Omega)$?

Closure of $C^\infty_0(\Omega)$ under $\|\ \|_{1,2}$.

"zero on the boundary"

Outline

$u(x,t) = U(x)\, e^{i\omega t}$

$\partial_t^2 u = \Delta u$

"bad"

Helmholtz

indefinite $\to \Delta u + k^2 u = 0$

definite $\to -\Delta u + k^2 u = 0$

"good Helmholtz"

Yukawa

screened Poisson

An Elliptic Model Problem

$\Delta = \operatorname{div} \operatorname{grad} u$

Let $\Omega \subset \mathbb{R}^n$ open, bounded, $f \in H^1(\Omega)$.

$$\begin{aligned} -\nabla \cdot \nabla u + u &= f(x) \quad (x \in \Omega), \\ u(x) &= 0 \qquad (x \in \partial\Omega). \end{aligned}$$

"Poisson eqn"

$-\nabla \cdot \nabla u = f$

Let $V := H_0^1(\Omega)$. Integration by parts? (Gauss's theorem applied to $a\boldsymbol{b}$):

$$\boxed{\int_\Omega \nabla a \cdot b + \int a \nabla \cdot \vec{b} = \int_\Omega \nabla \cdot (a\vec{b}) = \int_{\partial\Omega} \hat{n} \cdot (a\vec{b})}$$

Weak form?

$$\int -\nabla \cdot \nabla u \, v + \int u \, v = \int f \, v \qquad v \in H_0^1$$

$b = \nabla u$

$a = v$

$$\int_\Omega \nabla u \cdot \nabla v - \underbrace{\int_{\partial\Omega} \hat{n} \cdot (v \nabla u)}_{=0} + \int uv = \int fv$$

Motivation: Bilinear Forms and Functionals

$$\underbrace{\int_\Omega \nabla u \cdot \nabla v + \int_\Omega uv}_{=(u,v)_{H^1}} = \int fv. \qquad V = H^1_0$$

This is the weak form of the strong-form problem. The task is to find a $u \in V$ that satisfies this for all test functions $v \in V$.

Recast this in terms of bilinear forms and functionals:

$$a(u,v) = (\nabla u, \nabla v)_{L^2} + (u,v)_{L^2}$$

$$g(v) = (f,v)_{L^2}$$

Want $u \in V$: $\quad a(u,v) = g(v)$ for all $v \in V$

Dual Spaces and Functionals

$g(\alpha v + \beta w) = \alpha g(v) + \beta g(w)$

Bounded Linear Functional

Let $(V, \|\cdot\|)$ be a Banach space. A linear functional is a linear function $g : V \to \mathbb{R}$. It is bounded ($\Leftrightarrow$ continuous) if there exists a constant C so that $|g(v)| \leq C \|v\|$ for all $v \in V$.

Dual Space

Let $(V, \|\cdot\|)$ be a Banach space. Then the dual space V' is the space of bounded linear functionals on V.

Dual Space is Banach (cf. e.g. Trèves 1967)

V' is a Banach space with the dual norm

$$\|g\|_{V'} = \sup_{v \in V \setminus \{0\}} \frac{|g(v)|}{\|v\|_V}$$

Functionals in the Model Problem

Is g from the model problem a bounded functional? (In what space?)

$$\|g\|_{V'} = \sup_{v\in H^1\setminus\{0\}} \frac{(f,v)_{L^2}}{\|v\|_{H^1}} \overset{C\text{-}S}{\le} \frac{\|f\|_{L^2}\|v\|_{L^2}}{\|v\|_{L^2}+\|D_w v\|_{L^2}}$$

$$\le \frac{\|f\|_{L^2}\cancel{\|v\|}_{L^2}}{\cancel{\|v\|}_{L^2}}$$

That bound felt loose and wasteful. Can we do better?

$$\|f\|_{H^{-1}} = \underline{\sup_{v\in H^1\setminus\{0\}} \frac{(f,v)_{L^2}}{\|v\|_{H^1}}}$$

$$|g(v)| \le \|f\|_{H^{-1}}\,\|v\|_{H^1}$$

Riesz Representation Theorem (1/3)

Let V be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$.

Theorem (Riesz)

Let g be a bounded linear functional on V, i.e. $g \in V'$. Then there exists a unique $u \in V$ so that $g(v) = \langle u, v \rangle$ for all $v \in V$.

Let $g \in V'$. $\qquad N(\cdot)$: "null space"

Case I: $N(g) = V$. Pick $u = 0$, done.

Case II: $N(g) \neq V$. Let $w \in N(g)^\perp \setminus \{0\}$. $g(w) = \alpha \neq 0$

$$g\left(\frac{g(v)}{\alpha} w\right) = \frac{g(v)}{g(w)} \cdot g(w) = g(v)$$

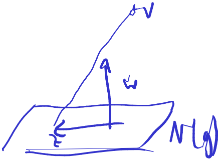
Let $v \in V$. Let $z := v - (g(v)/\alpha)\, w$. $\qquad g(z) = 0 \Leftrightarrow z \in N(g)$.

$\Rightarrow z \perp w \Leftrightarrow (z, w) = 0$.

Have $w \in N(g)^{\perp} \setminus \{0\}$, $\alpha = g(w) \neq 0$, and $z := v - (g(v)/\alpha)w \perp w$.

$$0 = (z, w) \iff \left(\frac{g(v)}{\alpha} w, w\right) = (v, w) \quad \text{For all } v$$

Multiply by $\frac{\alpha}{(w,w)}$



$$g(v) = \frac{\alpha}{(w,w)} \left(\frac{g(v)}{\alpha} w, w\right) = (v, w) \frac{\alpha}{(w,w)}$$

$$= \Big(v, \underbrace{\frac{\alpha}{(w,w)} w}_{u :=}\Big)$$

Riesz Representation Theorem: Proof (3/3)

Uniqueness of u?

Suppose $\tilde{u}$: $g(v) = (u, v) = (\tilde{u}, v)$

$\Leftrightarrow \quad (u - \tilde{u}, v) = 0 \quad \forall v$

Pick: $v = u - \tilde{u}$

$\Rightarrow \quad (u - \tilde{u}, u - \tilde{u}) = 0 = \|u - \tilde{u}\|^2 \Rightarrow u = \tilde{u}.$

Back to the Model Problem

$$
\begin{aligned}
a(u, v) &= \langle \nabla u, \nabla v \rangle_{L^2} + \langle u, v \rangle_{L^2} \\
g(v) &= \langle f, v \rangle_{L^2} \\
a(u, v) &= g(v)
\end{aligned}
$$

Have we learned anything about the solvability of this problem?