- HW 2
- Office hours moved
- Inst. message link

\[ u_t = \left( i u \right)^2 = (i u + \ldots)^2 \]

Dispersion / Dissipation

\[ e^{(kx - \omega t)} \]

\[ \text{Numerical O/D} \]

\[ z_{ji} = e^{(j kh_x - l \omega h_t)} \]

\[ P_{\text{Psi}} e^{l \omega t} = Q_{\text{Psi}} z_{\text{Psi}} \]

\[ \hat{\rho} (kh_x e^{-i\omega t}) \hat{z}_{\text{Psi}} = \hat{s} (kh_x) \hat{z} \cdot e \]

\[ s (k) \leq \text{large} \rightarrow \text{poorly resolved} \rightarrow \text{cost large} \rightarrow \text{well resolved} \]

\[ \lambda = \lambda (u, r) \]

\[ U^AV = D \]

\[ A_{\text{AV}} \text{AV} = \lambda (u, r) \]
Numerical Dispersion/Dissipation

Finite difference scheme $P_h u_{\ell+1} = Q_h u_\ell$ with symbol $s(k)$.

\[ z_{j,\ell} = z_0 e^{\log|s(\kappa)|\ell} e^{ik\left(jh_x - \frac{\phi(\kappa)}{kh_t}\right)h_t} \]

When is the scheme dissipative?

\[ |s(\kappa h_x)| < 1 \]

What is the phase speed?

\[ v_{ph} = \frac{-\phi(\kappa)}{kh_t} \]

Dispersion?

If $v_{ph}$ does not depend on $k$: all waves move with same speed; otherwise, dispersive
Dispersion/Dissipation Analysis of ETBS

Let $\lambda = ah_t/h_x$. Shown earlier: $s(kh_x) = 1 - \lambda(1 - e^{-ikh_x})$.

$$s(kh_x) = (1 - \lambda) + \lambda e^{-ikh_x}$$

Dissipation: \[ |s(u)| \text{ limits } \leq 1 \]

Smaller $\lambda \Rightarrow$ less dissipation per step.

$$e^{-i\omega(\kappa)h_t} = s(u) \approx 1 - \lambda(1 - e^{-i\kappa})$$
Dispersion/Dissipation Analysis of ETBS: Fine Grid

\[ e^{-i\omega(\kappa) t} = 1 - \lambda(1 - e^{-ikh_x}) \]

\[ S(\kappa) \approx 1 - \lambda + \lambda(1 - ik\kappa) = 1 - \lambda i\kappa \]

\[ e^{-i\omega(\kappa) t} \approx 1 - i\omega(\kappa) t \]

\[ 1 - i\omega(\kappa) t \approx 1 - i\omega(\kappa) t \]

\[ \omega(\kappa h_x) \approx \frac{\alpha h_x}{k h_t} \]

\[ v_{ph} = \frac{\omega(\kappa h_x)}{k h_t} \approx \alpha \]
Dispersion/Dissipation: Demo

- **Demo:** Experimenting with Dispersion and Dissipation [cleared]
- **Demo:** Dispersion and Dissipation [cleared]
Outline

Introduction

Finite Difference Methods for Time-Dependent Problems
  1D Advection
  Stability and Convergence
  Von Neumann Stability
  Dispersion and Dissipation
  A Glimpse of Parabolic PDEs

Finite Volume Methods for Hyperbolic Conservation Laws

Finite Element Methods for Elliptic Problems

Discontinuous Galerkin Methods for Hyperbolic Problems
Heat Equation

Heat equation \((D > 0)\):

\[
\begin{align*}
  u_t &= Du_{xx}, \quad (x, t) \in \mathbb{R} \times (0, \infty), \\
  u(x, 0) &= g(x) \quad x \in \mathbb{R}.
\end{align*}
\]

Fundamental solution \((g(x) = \delta(x))\):

\[
  u(x, t) = \frac{1}{\sqrt{4\pi t}} \ e^{-x^2/4t}
\]

Why is this a weird model?

\textit{No speed of propagation}
Schemes for the Heat Equation

Cook up some schemes for the heat equation.

Explicit Euler:

Implicit Euler: