Entropy condition:

\[ f'(c) > s > f'(u^+) \implies \text{"good shock"} \]

Convex flux: \[ f'(u^-) > f'(u^+) \]

Vanishing viscosity: \[ u \sim u^\epsilon \]

\[ (u^\epsilon)_t + f'(u^\epsilon)_x = \epsilon u_{xx} \]
Entropy-Flux Pairs

What are features of (physical) entropy?

Definition (Entropy/Entropy Flux)

An entropy $\eta(u)$ and an entropy flux $\psi(u)$ are functions so that $\eta$ is convex and

$$\eta(u)_t + \psi(u)_x = 0$$

for smooth solutions of the conservation law.
Finding Entropy-Flux Pairs

\[ \eta(u)_t + \psi(u)_x = 0. \] Find conditions on \( \eta \) and \( \psi \).

Come up with an entropy-flux pair for Burgers.

\[
\begin{align*}
\psi'(u) &= \eta'(u) \phi'(u) \\
\phi(u) &= \frac{u^2}{2} \\
\eta'(u) &= u^2 \\
\text{(one choice)} \quad \psi(u) &= 2u^3/3
\end{align*}
\]
Back to Vanishing Viscosity (1/2)

\[ u_t + f(u)_x = \varepsilon u_{xx} \]

What’s the evolution equation for the entropy?

\[ \eta'(u) u_t + \eta'(u) f'(u) u_x = \varepsilon \eta'''(u) u_{x} x \]

\[ \equiv \quad \eta'(u) u_t + \psi(u)_x = \varepsilon \left( \eta'''(u) u_x \right)_x - \varepsilon \eta''''(u) u_x^2 \]
Back to Vanishing Viscosity (2/2)

\[ \eta(u)_t + \psi(u)_x = \varepsilon(\eta'(u)u_x)_x - \varepsilon\eta''(u)u_x^2. \]

Integrate this over \([x_1, x_2] \times [t_1, t_2]\).

\[
\begin{align*}
\int_{t_1}^{t_2} \int_{x_1}^{x_2} \eta(u)_t + \psi(u)_x \, dx \, dt &= \int_{t_1}^{t_2} \left[ \eta'(u)u_x \right]_{x_1}^{x_2} \, dt \\
&\quad - \varepsilon \int_{t_1}^{t_2} \int_{x_1}^{x_2} \eta''(u)u_x^2 \, dx \, dt \\
&\geq 0
\end{align*}
\]

\[ \eta(u)_t + \psi(u)_x \leq 0 \]
The function \( u(x, t) \) is the entropy solution of the conservation law if for all convex entropy functions and corresponding entropy fluxes, the inequality

\[
\eta(u)_t + \psi(u)_x \leq 0
\]

is satisfied in the weak sense.
Entropy Solution vs Entropy Condition

Relate entropy solutions $\eta(u)_t + \psi(u)_x \leq 0$ back to the entropy condition.

Consider Burgers. $s_i = [u^i/2]/[u]$

$s_j = [2u^j/3]/[u^2] \leftarrow \text{R-H applied to entropy conservation}$

$s_j - s_i = [u]^2/[G(u_2+u_1)]$

$\begin{align*}
0 & \geq \left[ \int_{x_1}^{x_2} u^2 \, dx \right]_{t_1}^{t_1 + \Delta t} + \left[ \int_{t_1}^{t_1 + \Delta t} 2u^3/3 \, dx \right]_{x_1}^{x_2} \\
& = s_i \Delta t (u_{x_f}^2 - u_1^2) + \frac{2}{3} \Delta t (u_{x_f}^3 - u_2^3) + O(\Delta t^2) \\
& = \Delta t (u_{x_f}^2 - u_f^2) (s_i - s_2) + O(\Delta t^2) \\
& = \Delta t (u_{x_f}^2 - u_f^2) \left[ -\frac{1}{6} \frac{[u]^2}{u_2+u_1} \right] + O(\Delta t^2) \\
& = -\frac{1}{6} (u_{x_f}^2 - u_2^2)^3 \Delta t + O(\Delta t^2) \quad \forall \ u_2 \geq u_f$
\end{align*}$
Conservation of Entropy?

What can you say about conservation of entropy in time?