Saw numerically: interesting dispersion/dissipation behavior.

Want: theoretical understanding.

Consider \textit{linear, continuous} (not yet discrete) differential operators

\begin{align*}
L_1 u &= u_t + au_x, \\
L_2 u &= u_t - Du_{xx} + au_x \quad (D > 0) \\
L_3 u &= u_t + au_x - \mu u_{xxx}.
\end{align*}

What could we use as a prototype solution?
A Prototype Solution of the PDE

\[ u(x,t) = f(x - ct) \]

Observation: all these operators are diagonalized by complex exponentials. Come up with a ‘prototype complex exponential solution’.

\[ z(x,t) = z_0 e^{i(kx - wt)} \]

What type of function is this?

- \( k \in \mathbb{R} \): traveling wave w/speed \( c = \frac{w}{k} \)
- \( k \) imaginary: spatially decaying, damped wave
- \( \text{Im} w < 0 \): wave decaying in time
Wave-like Solutions of the PDE

\[ z(x, t) = z_0 e^{i(kx - \omega t)} \]

Observations in connection with \( L \)?

- \( Lz = \lambda(\omega, k) z \)
- \( Lz = 0 \) \( \Leftrightarrow \lambda(\omega, k) = 0 \)

What is the dispersion relation?

\( \lambda(\omega, k) = 0 \) is the d.r. for the PDE \( L \).
Picking Apart the Dispersion Relation \( \alpha \in \mathbb{R}, \beta \in \mathbb{R} \)

Consider \( \omega(k) = \alpha(k) + i\beta(k) \). Rewrite the wave solution with this.

\[
\begin{align*}
\psi(x,t) &= \psi_0 e^{i(kx - \omega(k)t)} \\
&= \psi_0 e^{i(kx - \alpha(k)t - i\beta(k)t)} \\
&= \psi_0 e^{\beta u(t)} e^{i(kx - \alpha(k)t)}
\end{align*}
\]

How can we recognize dissipation?

If \( \beta(u) < 0 \), we call the PDE dissipative.

What is the phase speed? How can we recognize dispersion?

- The phase speed is \( v_{ph}(k) = \frac{d\omega}{dk} \).
- If \( v_{ph}(k) \equiv \text{const.} \), \( \omega(k) \) is linear in \( k \), all waves move at the same speed.
- If that's not the case, we will call the PDE dispersive.
Dispersion Relation: Examples

In each case, find the dispersion relation and identify properties.

\[ L_1 u = u_t + au_x \]

\[ \lambda(u,k) = i(ak - \omega) = 0 \implies \omega = ak \]

no dissipation, no dispersion

\[ L_2 u = u_t - Du_{xx} + au_x \quad (D > 0) \]

\[ \lambda(u,k) = -i\omega + Oh^2 + iak = 0 \implies \omega = ak - iOh^2 \]

no dispersion, yes dissipation

\[ L_3 u = u_t + au_x - \mu u_{xxx} \]

\[ \lambda(u,k) = -i\omega + iak + i\mu k^3 = 0 \implies \omega = ak + \mu k^3 \]

no dissipation, yes dispersion
Goal: Want discrete finite difference scheme to match dissipation/dispersion behavior of continuous PDE.

Define a discrete wave-like function:

\[ z_{j,l} = e^{i(k_j h_x - \omega_l h_t)} \]

We want \( z \) to solve \( P_h z_{\ell+1} = Q_h z_\ell \). How can we connect the operators to the wave solution?

Assume \( P_h \) and \( Q_h \) are Toeplitz.
Theorem (Waves Diagonalize Toeplitz Operators)

Let $T$ be a Toeplitz operator. Then $T z_\ell = \lambda(k) z_\ell = \hat{t}(kh_x) z_\ell$.

\[
(T \xi_\ell)_j = \sum_m e_{m,j} t_{j-m} = \sum_m z_0 e^{i(km h_x - \omega \ell h_t)} t_{j-m}
\]

\[
= \left( \sum_m z_0 e^{i(k(m-j) h_x)} t_{j-m} \right) e^{i(kjh_x - \omega \ell h_t)}
\]

\[
= \left( \sum_{m'} e^{-i km' h_x} t_{m'} \right) \xi_j \ell
\]

\[
= \hat{t}(kh_x) \xi_j \ell = \lambda \xi_j \ell
\]
Waves and Two-Level Schemes

Since $P_h$ and $Q_h$ are Toeplitz, we must have

$$P_h z_{\ell+1} = \lambda P(k) z_{\ell+1}, \quad Q_h z_{\ell} = \lambda Q(k) z_{\ell}.$$ 

What does that mean?

\[
\begin{align*}
\lambda_p(k) \tilde{z}_{\ell+1} &= \lambda_Q(k) \tilde{z}_{\ell} \\
\lambda_p(k) e^{-i\omega h t} \tilde{z}_{\ell} &= \lambda_Q(k) \tilde{z}_{\ell} \\
e^{-i\omega h t} &= \frac{\lambda Q(k)}{\lambda P(k)} = \frac{q(k)}{p(k)} = s(k) = s(kh_x)
\end{align*}
\]

Seen before? 

\[ \text{un stability} \]
So $z_\ell$ is a solution of the finite difference scheme if $\omega = \omega(kh_x)$ satisfies

$$e^{-i\omega(\kappa)h_t} = s(\kappa),$$

where we let $\kappa = kh_x$. Interpret $\kappa$.

Let $s(\kappa) = |s(\kappa)| e^{i\varphi(\kappa)} = e^{\log|s(\kappa)|+i\varphi(\kappa)}$. $\omega(\kappa)$?

$$\omega = \frac{-\varphi(\kappa) + i\log|s(\kappa)|}{h_t}$$

PPW = $\frac{1}{\kappa}$

the number of wavelengths per point
Discrete Dispersion Relation (2/2)

\[ \omega(\kappa) = \frac{-\varphi(\kappa) + i \log |s(\kappa)|}{h_t}. \]

Plug that into the wave-like solution:

\[ z_{j,l} = z_0 e^{i(k_j h_x - \omega(\kappa) l h_t)} \]
\[ = z_0 \exp \left( i \left( k_j h_x - \frac{\varphi(\kappa) + i \log |s(\kappa)|}{h_t} \right) l h_t \right) \]
\[ = z_0 \exp \left( \log |s(\kappa)| l \right) \exp \left( i k_l \left( j h_x - \frac{i \log |s(\kappa)|}{h_t} \right) l h_t \right) \]

Criterion for stability?

\[ |s(\kappa)| \leq 1 \quad (\text{as with } uN) \]
Numerical Dispersion/Dissipation

Finite difference scheme \( P_h \mathbf{u}_{\ell+1} = Q_h \mathbf{u}_\ell \) with symbol \( s(k) \).

\[
z_{j,\ell} = z_0 e^{\log |s(\kappa)|\ell} e^{ik\left(jh_x - \frac{-\varphi(\kappa)}{kh_t} \ell h_t\right)}
\]

When is the scheme **dissipative**?

*If \( |s(kh_x)| < 1 \), then the scheme is dissipative*.

*with factor \( |s(kh_x)| \).*

What is the **phase speed**?

\[v_{ph} = \frac{-\varphi(kh_x)}{kh_t}\]

Dispersion?

*If \( v_{ph} \equiv \text{const} \), the scheme is non-dispersive.*

*Otherwise, it is.*
Dispersion/Dissipation Analysis of ETBS

Let $\lambda = ah_t / h_x$. Shown earlier: $s(kh_x) = 1 - \lambda(1 - e^{-ikh_x})$. 