Numerical Methods for Partial Differential Equations

CS555 / MATH552 / CSE510

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Outline

Introduction
- Notes
- Notes (unfilled, with empty boxes)
- About the Class
- Classifcation of PDEs
- Preliminaries: Differencing
- Interpolation Error Estimates (reference)

wave-like
diffusion-like

Finite Difference Methods for Time-Dependent Problems

1D + time

Finite Volume Methods for Hyperbolic Conservation Laws

Finite Element Methods for Elliptic Problems

2D/3D often, time-indep

Discontinuous Galerkin Methods for Hyperbolic Problems

Outline

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What's the point of this class?

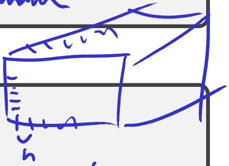
→ solid mechanics

PDEs describe lots of things in nature:

- Fluid flow → Navier-Stokes
- electromagnetics → Maxwell's equ's
- plasmas → MHD
- rarefied gases → Vlasov/Boltzmann

Idea: Use them to

- science / model / discovery
- simplified models
- use predictions for design



$h \to \frac{h}{2}$

CEESD

Survey

- Home dept
- Degree pursued
- Longest program ever written
  - in Python?
- Research area

Class web page

https://bit.ly/numpde-s22

- Book Draft
- Notes, Class Outline
- Assignments (submission and return)
- Piazza
- Grading Policies/Syllabus
- Video
- Scribbles
- Demos (binder)

Sources for these Notes

- Adler, James, Hans De Sterck, Scott MacLachlan, and Luke N. Olson. *Numerical Partial Differential Equations*, 2022. (draft)
- Strikwerda, John C. *Finite Difference Schemes and Partial Differential Equations*, Second Edition. Other Titles in Applied Mathematics. Society for Industrial and Applied Mathematics, 2004.
- LeVeque, Randall J. *Numerical Methods for Conservation Laws*. 2nd ed. Birkhäuser Basel, 1992.
- Braess, Dietrich. *Finite Elements: Theory, Fast Solvers, and Applications in Solid Mechanics*. Cambridge University Press, 2007.
- Shu, Chi-Wang. *Lecture Notes for AM257*, Brown University, Fall 2006.
- Heuveline, Vincent. *Lecture Notes for "Numerik für PDEs"*. Universität Karlsruhe, Summer 2005.
- Various prior bits of material by Luke Olson and Stephen Bond.

Open Source <3

These notes (and the accompanying demos) are open-source!

Bug reports and pull requests welcome:
https://github.com/inducer/numpde-notes

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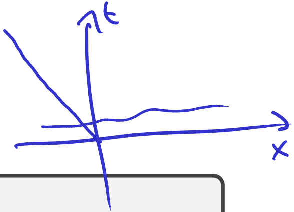
Outline

PDEs: Example I

$$u(t,x) = f(t+x)$$



What does this do? $\partial_t u = \partial_x u$

$$\partial_t u = f'(t+x)$$

$$\partial_x u = +f'(t+x)$$

$x = -t$

- "one-way wave equation"
- "advection equation"

$f(u) = -u$

$$u_t - u_x = 0 \rightsquigarrow u_t + f(u)_x = 0$$

$f'(u) \rightsquigarrow$ "char. speed"

↳ "cons. law"

$$u_t + (au)_x = 0$$

PDEs: Example II

Laplace

What does this do? $\partial_x^2 u + \partial_y^2 u = 0$

- second derivative ≃ "bendiness"
- bendiness in x/y need to add up to 0
- minima/maxima are not allowed

Some good questions

→ info only travels forward in t

- What is a time-like variable? (Variables labeled t?)
- What if there are boundaries?
  - In space?
  - In time?
- Existence and Uniqueness of Solutions?
  - Depends on where we look (the *function space*)
  - In the case of the two examples? (if there are no boundaries?)

$u(t, x, y)$ $x, y \in \mathbb{R}^2$

BC →

t

IC

Some general takeaways:

- use commonsense
- use phys. intuition

PDEs: An Unhelpfully Broad Problem Statement

Looking for $u : \Omega \to R^n$ where $\Omega \subseteq \mathbb{R}^d$ so that $u \in V$ and

$$F(u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \ldots, x, y, \ldots) = 0$$

$u_{xy} = 15$

Notation

Used as convenient:

$$u_x = \partial_x u = \frac{\partial u}{\partial x}$$

Properties of PDEs

What is the order of the PDE?

The highest occurring derivative degree, summing over dimensions.

When is the PDE linear?

If u, v are solutions, then $\alpha u + \beta v$ is one as well. $\alpha, \beta \in \mathbb{C}, \mathbb{R}$

When is the PDE quasilinear?

The dep. in F on the highest partial order derivative is linear in u.

When is the PDE semilinear?

If it is quasilinear and if the highest order terms have coeffs only depending on space.

Examples: Order, Linearity?

$$(xu^2)u_{xx} + (u_x + y)u_{yy} + u_x^3 + yu_y = f$$

Quasilinear, order 2

$$(x + y + z)u_x + (z^2)u_y + (\sin x)u_z = f$$

Semilinear, order 1

Properties of Domains