Announcements

- In person next week (hopefully?)
    - 3025 CIF
- discussion invites
- HW1

Examples: Order, Linearity?

$$(xu^2)u_{xx} + (u_x + y)u_{yy} + u_x^3 + yu_y = f$$

quasilinear, 2nd order

$$(x + y + z)u_x + (z^2)u_y + (\sin x)u_z = f$$

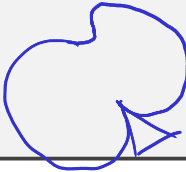
semilinear, 1st order

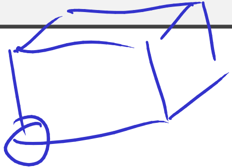
Properties of Domains

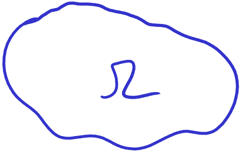
$u(t, x)$ $\quad x \in \Omega \subset \mathbb{R}^d$

$t \in [0, T]$

- Domain impact on existence of solution?
  - corners
  - reentrant corners
  - cusps



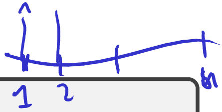




Function Spaces: Examples

↳ vector

$\mathbb{R}^n$



Name some function spaces with their norms.

$C^0(\Omega)$: continuous

$C^k(\Omega)$: k-times continuously differentiable

$C^{0,\alpha}(\Omega)$: $\|f\|_\alpha = \|f\|_\infty + \sup_{x \neq y} \frac{\|f(x)-f(y)\|}{\|x-y\|}$

↳ max abs value

$C_L(\Omega)$: "Lipschitz cont" $\|f(x)-f(y)\| \leq C\|x-y\|$

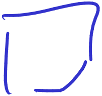
$L^2(\Omega) = \{ u : \Omega \to \mathbb{R} : \int_\Omega |u|^2 \, dx < \infty \}$ $\|f\|_2 = \sqrt{\ldots}$

$H^1(\Omega) = \|f\|_2 + \|f'\|_2$

May *also* influence existence/uniqueness of solutions!

weak derivative

Solving PDEs



$u(t,x) = U(x) \cdot V(t)$

Closed-form solutions:

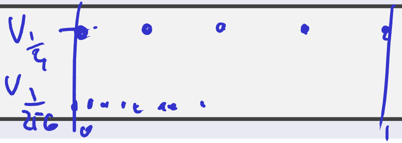
- If separation of variables applies to the domain: good luck with your ODE
- If not: Good luck! $\to$ Numerics

General Idea (that we will follow some of the time)

- Pick $V_h \subseteq V$ finite-dimensional
  - h is often a *mesh spacing*
- Approximate u through $u_h \in V_h$
- Show: $u_h \to u$ (in some sense) as $h \to 0$

Example

$u(x) = \sin(x)$



About grand big unifying theories

Is there a grand big unifying theory of PDEs?

NO

Collect some stamps

2D second-order linear

$$a(x,y)u_{xx}+2b(x,y)u_{xy}+c(x,y)u_{yy}+d(x,y)u_x+e(x,y)u_y+f(x,y)u=g(x,y)$$

Discriminant value	Kind	Example
$b^2-ac<0$	Elliptic	Laplace $u_{xx}+u_{yy}=0$
$b^2-ac=0$	Parabolic	Heat $u_t=u_{xx}$
$b^2-ac>0$	Hyperbolic	Wave $u_{tt}=u_{xx}$

Where do these names come from?

search for characteristic curves

↳ see lecture notes by Hogg

PDE Classification in Other Cases

Scalar first order PDEs?

Hyperbolic

First order systems of PDEs?

all types (ell, par, hyp) are possible,
see Hogg for classification

Classification in higher dimensions

$\mathbb{R}^n$

$$Lu := \sum_{i=1}^{d}\sum_{j=1}^{d} a_{i,j}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \text{lower order terms}$$

Consider the matrix $A(x) = (a_{ij}(x))_{i,j}$. May assume A symmetric. Why?

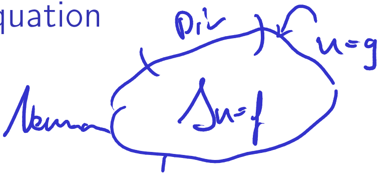
Schwarz's theorem

What cases can arise for the eigenvalues?

(λ_j) eigenvalues of A

$\exists j: \lambda_j = 0$	parabolic case
λ_j all have same sign	elliptic case
all but one have same sign	hyp
more than one with diff. signs	ultra-hyperbolic

Elliptic PDE: Laplace/Poisson Equation



$$\triangle u = \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2} = \nabla \cdot \nabla u(x) \overset{\text{2D}}{=} u_{xx} + u_{yy} = f(x) \quad (x \in \Omega)$$

Called Laplace equation if $f = 0$. With Dirichlet boundary condition

$$u(x) = g(x) \qquad (x \in \partial\Omega).$$

Demo: Elliptic PDE Illustrating the Maximum Principle [cleared]

$\partial_n u = g$ ("Neumann")

$\alpha u + \beta \partial_n u = g$ ("Robin")

Elliptic PDEs: Singular Solution

$\mathcal{L}u = \Delta u = \underline{\qquad}$

Fundamental

$\mathcal{L}u = \Delta u = \delta$

Demo: Elliptic PDE Radially Symmetric Singular Solution [cleared]

$\Delta G = \delta$

Given $G(x) = C\log(|x|)$ as the free-space Green's function, can we construct the solution to the PDE with a more general f?

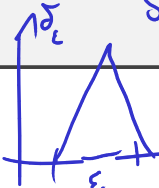
$\to \Delta u = f$

$$u(x) = (G * f)(x) = \int_{\mathbb{R}^2} G(x-y) f(y)\,dy$$

$\Rightarrow \Delta u = f$

What can we learn from this?

Solution is globally coupled



$\int \delta_\varepsilon = 1$

$\int \Delta u \varphi = \int \delta \varphi$

Elliptic PDEs: Justifying the Singular Solution

$$u(x) = (G * f)(x) = \int_{\mathbb{R}^d} G(x-y)f(y)dy$$

$$\Delta G(x-y) = \delta(x-y)$$

Why?

$$\Delta u(x) = \Delta \int G(x-y) f(y)\, dy$$
$$= \int \Delta G(x-y) f(y)\, dy$$
$$= \int \delta(x-y) f(y)\, dy = f(x)$$

$$\int \delta .$$

Parabolic PDE: Heat Equation · Separation of Variables

Cap. $u_{xx} + u_{yy} = 0$

Wave $u_{tt} = u_{xx}$

$$u_t = u_{xx} \qquad ((x,t) \in [0,1] \times [0,T])$$

$$u(x,0) = g(x) \qquad (x \in [0,1])$$

$$u(0,t) = u(1,t) = 0 \qquad (t \in [0,T])$$

$u(x,t) = v(t) \cdot w(x)$

Plug into PDE: $v'(t) \cdot w(x) = v(t) \cdot w''(x)$

$$\frac{v'(t)}{v(t)} = C = \frac{w''(x)}{w(x)}$$

$v'(t) = C \cdot v(t)$

$v(t) = \exp(-m^2 \pi^2 t)$

$w''(x) = C \cdot w(x)$

$w(x) = \alpha \cdot \sin(m \pi x)$

$\rightsquigarrow C = -m^2 \pi^2$

Parabolic PDE: Solution Behavior

Demo: Parabolic PDE [cleared] What can we learn from analytic and numerical solution?

- "washes out" the solution
- fund. solution


- data becomes smoother

Hyperbolic PDE: Wave Equation

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \qquad ((x,t) \in \mathbb{R} \times [0,T]) \\ u(x,0) &= g(x) \qquad (x \in \mathbb{R}) \end{aligned}$$

with $g(x) = \sin(\pi x)$.

Is this problem well-posed?

Can be rewritten in conservation law form: