

Hyperbolic PDE: Wave Equation

CS555

$u_t + a u_x = 0$
 $u_{tt} = c^2 u_{xx} \quad ((x, t) \in \mathbb{R} \times [0, T])$
 $u(x, 0) = g(x) \quad (x \in \mathbb{R})$
 with $g(x) = \sin(\pi x)$.

$$\frac{dy}{dt} = f(y)$$

Is this problem well-posed?

No, missing IC on u_t

$$u_t(x, 0) = 0 \quad (x \in \mathbb{R})$$

Can be rewritten in **conservation law** form:

$$\rightarrow \vec{q}_t + \nabla \cdot F(q) = s(x,t) \quad q(x,t)$$

Hyperbolic Conservation Laws

$$q \in \mathbb{R}^h$$

$$x \in \mathbb{R}^d$$

$$\vec{q}_t(\vec{x}, t) + \nabla \cdot \mathbf{F}(\vec{q}(\vec{x}, t)) = s(x)$$



Why is this called a conservation law?

$$\int_{\Omega'} q_t + \nabla \cdot \mathbf{F}(q) = 0 \quad \rightarrow s(x) = 0$$

$$\partial_t \int_{\Omega'} q + \int_{\partial\Omega'} \nabla \cdot \mathbf{F}(q) = 0$$

$$\partial_t \int_{\Omega'} q + \int_{\partial\Omega'} \hat{n} \cdot \mathbf{F}(q) = 0$$

$F : ? \rightarrow ?$

mass of 'gas'
 ∂_t

electromagnetics
fluid flow

$$F : \mathbb{R}^h \rightarrow \mathbb{R}^{h \times d}$$

Wave Equation as a Conservation Law

$$u_{tt} = c^2 u_{xx}$$

Rewrite the wave equation in conservation law form:

$$\begin{cases} u_t = c v_x \\ v_t = c u_x \end{cases}$$
$$\hookrightarrow u_{tt} = (c v_x)_t = c (v_t)_x = c^2 u_{xx}$$

$$\vec{q} = \begin{pmatrix} u \\ v \end{pmatrix} \quad F(q) = \begin{bmatrix} 0 & -c \\ -c & 0 \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \quad \begin{array}{l} z = y' \\ y'' = f(y, y') \end{array}$$
$$q_t + \text{div } F(q) = 0 \quad \hookrightarrow \begin{pmatrix} y \\ z \end{pmatrix}' = \begin{pmatrix} z \\ -f(y, z) \end{pmatrix}$$

Solving Conservation Laws

Solve

$$u_t = v_x$$

$$v_t = u_x.$$

A

$$q_t + \begin{pmatrix} 0 & -c \\ -c & 0 \end{pmatrix} q_x = 0$$

$$\tilde{q} := V^{-1} q$$

$$D = V^{-1} A V = \text{diag}(c, -c)$$

$$\tilde{q}_t + V^{-1} A V \tilde{q}_x = 0$$

$$\tilde{q}_t + D \tilde{q}_x = 0$$

$$\begin{aligned} w_t + c w_x &= 0 \\ y_t - c y_x &= 0 \end{aligned}$$

$$q = (u, v)$$

$$\tilde{q} = (w, y)$$

Hyperbolic: Solution Properties

Properties of the solution for hyperbolic equations:

- have conserved quantities
(e.g. energy, cf. HW1)
- For linear conservation laws, smoothness of IC+BC data determines smoothness of solution for all time
(nonlinear, maybe not)
- Diagonalization of flux Jacobian yields characteristic speeds

Outline

Introduction

Notes

Notes (unfilled, with empty boxes)

About the Class

Classification of PDEs

Preliminaries: Differencing

Interpolation Error Estimates (reference)

Finite Difference Methods for Time-Dependent Problems

Finite Volume Methods for Hyperbolic Conservation Laws

Finite Element Methods for Elliptic Problems

Discontinuous Galerkin Methods for Hyperbolic Problems

Interpolation and Vandermonde Matrices

$\{x_1, \dots, x_n\}$
 $\{\phi_1, \dots, \phi_n\}$

$$f(x_i) \approx \sum_{i=1}^N \alpha_i \phi_i(x_i) = p_{N-1}(x_i)$$

$$V_{ij} = (\phi_j(x_i))_{ij}$$

$$V\vec{\alpha} = f(\vec{x}) \Leftrightarrow \vec{\alpha} = V^{-1}f(\vec{x})$$

$$f'(x_i) \approx \sum_{i=1}^N \alpha_i \phi'_i(x_i) = p'_{N-1}(x_i) \quad V'_{ij} = (\phi'_j(x_i))_{ij}$$

$$V'\vec{\alpha} = p'_{N-1}(\vec{x}) \approx f'(\vec{x})$$

$$f'(\vec{x}) = V'V^{-1}f(\vec{x})$$

Numerical Differentiation: How?

How can we take derivatives numerically?



Demo: Taking Derivatives with Vandermonde Matrices [cleared]