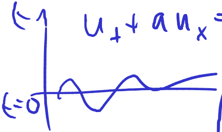
Hyperbolic PDE: Wave Equation

CS555

$u_t + a u_x = 0$



$$u_{tt} = c^2 u_{xx} \qquad ((x,t) \in \mathbb{R} \times [0,T])$$

$$u(x,0) = g(x) \qquad (x \in \mathbb{R})$$

$$\frac{dy}{dt} = f(y)$$

with $g(x) = \sin(\pi x)$.

Is this problem well-posed?

No, missing IC on u_t

$$u_t(x,0) = 0 \quad (x \in \mathbb{R})$$

Can be rewritten in conservation law form:

$$\to \vec{q}_t + \nabla \cdot F(q) = s(x,t) \qquad q(x,t)$$

Hyperbolic Conservation Laws

$q \in \mathbb{R}^n$

$x \in \mathbb{R}^d$

$$\vec{q}_t(\vec{x}, t) + \nabla \cdot \boldsymbol{F}(\vec{q}(\vec{x}, t)) = s(x)$$

Why is this called a conservation law?

$\hookrightarrow s(x) = 0$

$$\int_{\Omega'} q_t + \nabla \cdot F(q) = 0$$

$$\partial_t \int_{\Omega'} q + \int_{\Omega'} \nabla \cdot F(q) = 0$$

$$\underbrace{\partial_t \int_{\Omega'} q}_{\partial_t \text{ mass of 'gas'}} + \int_{\partial \Omega'} \hat{n} \cdot F(q) = 0$$

$\hookrightarrow$ electromagnetics, fluid flow

F :? $\to$?

$$F : \mathbb{R}^n \to \mathbb{R}^{n \times d}$$

Wave Equation as a Conservation Law

$$u_{tt} = c^2 u_{xx}$$

Rewrite the wave equation in conservation law form:

$$\begin{cases} u_t = c v_x \\ v_t = c u_x \end{cases}$$

$$\hookrightarrow u_{tt} = (c v_x)_t = c (v_t)_x = c^2 u_{xx}$$

$$\vec{q} = \begin{pmatrix} u \\ v \end{pmatrix} \qquad F(q) = \begin{bmatrix} 0 & -c \\ -c & 0 \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

q_t + div F(q) =0

$$z = y' \qquad y'' = f(y, y')$$

$$\hookrightarrow \begin{pmatrix} y \\ z \end{pmatrix}' = \begin{pmatrix} z \\ f(y,z) \end{pmatrix}$$

Solving Conservation Laws

Solve

$$u_t = v_x$$
$$v_t = u_x.$$

$$q_t + \overset{A}{\begin{pmatrix} 0 & -c \\ -c & 0 \end{pmatrix}} q_x = 0$$

$$\tilde{q} := V^{-1} q \qquad D = V^{-1} A V = \operatorname{diag}(c, -c)$$

$$\tilde{q}_t + V^{-1} A V \tilde{q}_x = 0$$
$$\tilde{q}_t + D \tilde{q}_x = 0$$

$$w_t + c w_x = 0$$
$$y_t - c y_x = 0$$

$$q = (u, v) \qquad \tilde{q} = (w, y)$$

Demo: Hyperbolic PDE [cleared]

Hyperbolic: Solution Properties

Properties of the solution for hyperbolic equations:

- have conserved quantities
  (e.g. energy, cf. HW1)
- For linear conservation laws, smoothness of IC+BC data
  determines smoothness of solution for all time
  (nonlinear, maybe not)
- Diagonalization of flux Jacobian yields characteristic
  speeds

Outline

Interpolation and Vandermonde Matrices

{x_1, ..., x_n}
{phi_1, ..., phi_n}

$$f(x_i) \approx \sum_{i=1}^{N} \alpha_i \phi_i(x_i) = p_{N-1}(x_i)$$

$$V_{ij} = (\phi_j(x_i))_{ij}$$

$$V\vec{\alpha} = f(\vec{x}) \Leftrightarrow \vec{\alpha} = V^{-1} f(\vec{x})$$

$$f'(x_i) \approx \sum_{i=1}^{N} \alpha_i \phi_i'(x_i) = p_{N-1}'(x_i) \qquad V_{ij}' = (\phi_j'(x_i))_{ij}$$

$$V'\vec{\alpha} = p_{N-1}(\vec{x}) \approx f'(\vec{x}) \qquad f'(\vec{x}) = V'V^{-1} f(\vec{x})$$

Numerical Differentiation: How?

How can we take derivatives numerically?

Demo: Taking Derivatives with Vandermonde Matrices [cleared]