- HW1 due today
- Instant message

Stability of ETBS (3/3)

$u_t + a u_x = 0$

$\|(P_h^{-1} Q_h)^\ell P_h^{-1}\| \le C$

$\Rightarrow \|P_h^{-1} Q_h\| \le 1 \quad \|P_h^{-1}\| \le C$

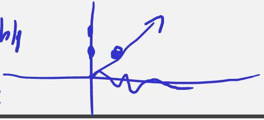
$\|Q_h\| \quad \sigma(Q_h^T Q_h)$

Summarize ETBS stability: $(a > 0)$

$$\lambda = \frac{a h_t}{h_x} \le 1 \iff h_t \le \frac{h_x}{a} \lambda$$

"downwind" differencing: bad / unstable

"upwind differencing": good / unstable



$\|P_h^{-1}\| \; \|Q_h\|$

Comments?

Yuck, cumbersome

Outline

Discrete (Space) Fourier Transform

Assume $\boldsymbol{x}$ infinitely long. Define:

$$\hat{\boldsymbol{x}}(\theta) = \sum_k x_k e^{-i\theta k} \qquad (\theta \in (-\pi, \pi))$$

When is this well-defined?

$$\left|\hat{x}(\theta)\right| = \left|\sum_k x_k e^{-i\theta k}\right| \leq \sum_k |x_k| < \infty$$

- series with (x_k) is absolutely convergent
- $\|\vec{x}\|_1 < \infty$

Inverting the Fourier Transform

To recover $\boldsymbol{x}$:

$$x_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{\boldsymbol{x}}(\theta) e^{i\theta k} d\theta.$$

Proof?

$$x_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_j x_j e^{-i\theta j} e^{\theta k} \, d\theta = \frac{1}{2\pi} \sum_j x_j \underbrace{\int_{-\pi}^{\pi} e^{i\theta(k-j)} d\theta}_{2\pi\delta_{j,k}}$$

$$= \sum_j x_j \delta_{jk} = x_k$$

Getting to L^2

- Fourier Transform well defined for $\boldsymbol{x} \in \ell^1$.
- Problem: We care about L^2, not ℓ^1.

Theorem (Parseval)

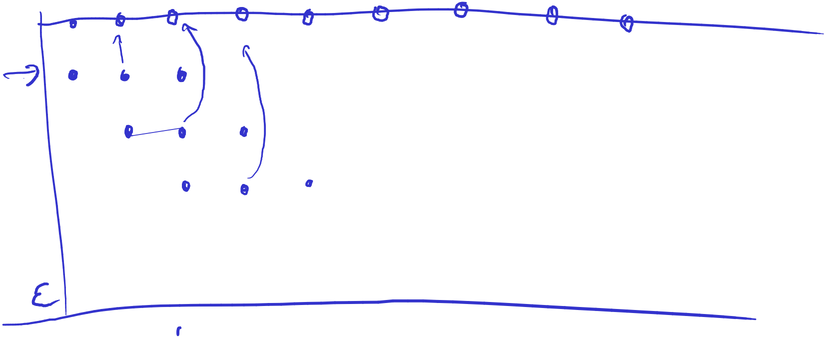
If $\|\boldsymbol{x}\|_2 < \infty$*, then*

$$\|\boldsymbol{x}\|_2^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{\boldsymbol{x}}(\theta)|^2 \, d\theta < \infty.$$

ℓ^2, L^2: $\|\hat{x}\|_{L^2(-\pi,\pi)}$

Impact?

can extend def. of FT to L^2.



"convolutions"

Toeplitz Operators

Definition (Toeplitz Operator)

An operator T is a Toeplitz operator if $(T\boldsymbol{x})_j = \sum_k x_k p_{j-k}$. In this case, $\boldsymbol{p}$ is called the Toeplitz vector.

Example: ETCS

Let $\lambda = ah_t/2h_x$. Then

$$u_{k,\ell+1} = \lambda u_{k-1,\ell} + u_{k,\ell} - \lambda u_{k+1,\ell}.$$

Is ETCS Toeplitz?

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$$(P_h \boldsymbol{u}_{\ell+1})_j = u_{j,\ell+1} \stackrel{!}{=} \textstyle\sum_k u_{k,\ell+1} p_{j-k}$$

$$p_{j-k} = \begin{cases} 1 & \text{if } k=j \\ 0 & \text{otherwise} \end{cases} \qquad p_m = \delta_{0,m}$$

$$\to (Q_h \boldsymbol{u}_\ell)_j = \lambda u_{k-1,\ell} + u_{k,l} - \lambda u_{k+1,\ell} \stackrel{!}{=} \textstyle\sum_k u_{k,\ell} q_{j-k}$$

$$q_{j-k} = \begin{cases} \lambda & \text{if } k=j-1 \\ 1 & \text{if } k=j \\ -\lambda & \text{if } k=j+1 \end{cases} \qquad q_m = \begin{cases} \lambda & m=1 \\ 1 & \text{if } m=0 \\ -\lambda & \text{if } m=-1 \\ 0 & \text{otherwise} \end{cases}$$

$j-k=m$

Fourier Transforms of Toeplitz Operators (1/3)

$$y_j = \sum_k x_k p_{j-k}$$

$$\hat{y}(\theta) = \sum_j \sum_k x_k p_{j-k} e^{-i\theta j}$$

$$= \sum_j \sum_k \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{x}(\varphi) e^{i\varphi k} d\varphi \right) p_{j-k} e^{-i\theta j}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{x}(\varphi) \sum_j \sum_k e^{i\varphi k} p_{j-k} e^{-i\theta j} \underbrace{e^{-i\varphi j} e^{i\varphi j}}_{1} d\varphi$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{x}(\varphi) \sum_j \underbrace{\left(\sum_k e^{i\varphi(k-j)} p_{j-k} \right)}_{\text{"}\hat{p}(\varphi)\text{"}} e^{i(\varphi-\theta)j} d\varphi$$

Fourier Transforms of Toeplitz Operators (2/3)

$$\hat{\boldsymbol{y}}(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{\boldsymbol{x}}(\varphi) \sum_j \overbrace{\left(\sum_k e^{i\varphi(k-j)} p_{j-k} \right)}^{\hat{p}(\varphi)} e^{i(\varphi-\theta)j} d\varphi.$$

$$\sum_k e^{i\varphi(k-j)} p_{j-k} = \sum_k e^{-i\varphi(j-k)} p_{j-k} = \sum_\ell e^{-i\varphi\ell} p_\ell = \hat{p}(\varphi)$$

$\ell = j-k$

Fourier Transforms of Toeplitz Operators (3/3)

$$\hat{\boldsymbol{y}}(\theta) = \int_{-\pi}^{\pi} \hat{\boldsymbol{x}}(\varphi)\hat{\boldsymbol{p}}(\varphi)\frac{1}{2\pi}\sum_{j} e^{i(\varphi-\theta)j}\, d\varphi.$$

$$\underline{w_j = \frac{1}{2\pi} e^{i\varphi j}}$$

Then $\hat{w}(\theta) = \frac{1}{2\pi} \sum_k e^{i(\varphi-\theta)k}$.

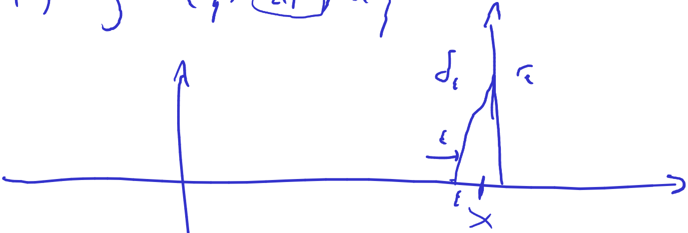
$$\Rightarrow \hat{y}(\theta) = \int_{-\pi}^{\pi} \hat{x}(\varphi)\, \hat{p}(\varphi)\, \hat{w}(\theta)\, d\varphi$$

To determine $\hat{w}(\theta)$

$$\frac{1}{2\pi} e^{i\varphi j} = w_j = \frac{1}{2\pi} \int \underbrace{\hat{w}(\theta)}_{\delta(\varphi-\theta)} e^{i\theta j}\, d\theta$$

$$\Rightarrow \hat{y}(\theta) = \hat{x}(\theta)\, \hat{p}(\theta).$$

$$u(x) \Leftarrow \int u(\xi)\, \underbrace{\delta_\varepsilon(\xi - x)}\, d\xi$$



$$\int \delta_\varepsilon = 1$$

"distribution"

Fourier Transforms of Inverse Toeplitz Operators

Fourier transform $P_h^{-1} Q_h \boldsymbol{y}$?

$$\frac{\hat{q}(\theta)}{\hat{p}(\theta)} \hat{y}(\theta)$$

Bounding the Operator Norm

Bound $\left\|P_h^{-1} Q_h\right\|_2^2$ using Fourier:

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Is the upper bound attained?