

- HW1 due today
- Instant message

Stability of ETBS (3/3)

Summarize ETBS stability:

$(a > 0)$

$$u_t + a u_x = 0$$

$$\left[\begin{aligned} \|P_n^{-1} Q_n\| \|P_n\| &\leq c \\ \Rightarrow \|P_n^{-1} Q_n\| &\leq c \|P_n\| \\ \|Q_n\| &= o(Q_n^T Q_n) \end{aligned} \right.$$

$$\lambda = \frac{a h_t}{h_x} \leq 1 \quad (\Leftrightarrow) \quad h_t \leq \frac{h_x}{a}$$

"downwind" differencing: bad/unstable

"upwind differencing": good/unstable



Comments?

$$\|P_n^{-1}\| \|Q_n\|$$

Yuck, cumbersome

Outline

Introduction

Finite Difference Methods for Time-Dependent Problems

1D Advection

Stability and Convergence

Von Neumann Stability

Dispersion and Dissipation

A Glimpse of Parabolic PDEs

Finite Volume Methods for Hyperbolic Conservation Laws

Finite Element Methods for Elliptic Problems

Discontinuous Galerkin Methods for Hyperbolic Problems

Discrete (Space) Fourier Transform

Assume x infinitely long. Define:

$$\hat{x}(\theta) = \sum_k x_k e^{-i\theta k} \quad (\theta \in (-\pi, \pi))$$

When is this well-defined?

$$|\hat{x}(\theta)| = \left| \sum_k x_k e^{-i\theta k} \right| \leq \sum_k |x_k| < \infty$$

- series with (x_n) is absolutely convergent
- $\|\vec{x}\|_1 < \infty$

Inverting the Fourier Transform

To recover \mathbf{x} :

$$x_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{\mathbf{x}}(\theta) e^{i\theta k} d\theta.$$

Proof?

$$\begin{aligned} X_k &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_i x_j e^{-i\theta j} e^{i\theta k} d\theta = \frac{1}{2\pi} \sum_j x_j \underbrace{\int_{-\pi}^{\pi} e^{i\theta(k-j)} d\theta}_{2\pi \delta_{jk}} \\ &= \sum_j x_j \delta_{jk} = x_k \end{aligned}$$

Getting to L^2

- ▶ Fourier Transform well defined for $\mathbf{x} \in \ell^1$.
- ▶ Problem: We care about L^2 , not ℓ^1 .

Theorem (Parseval)

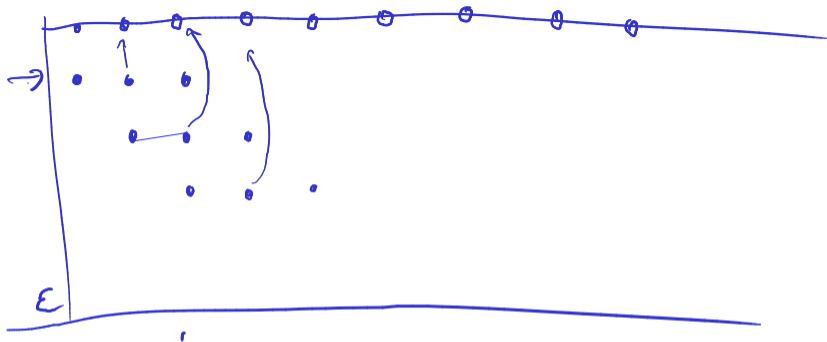
If $\|\mathbf{x}\|_2 < \infty$, then

$$\|\mathbf{x}\|_2^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{\mathbf{x}}(\theta)|^2 d\theta < \infty.$$

L^2 , $L^2: \|\hat{\mathbf{x}}\|_{C(-\pi, \pi)}$

Impact?

can extend def. of FT to L^2 .



"convolutions"

Toeplitz Operators

Definition (Toeplitz Operator)

An operator T is a **Toeplitz operator** if $(T\mathbf{x})_j = \sum_k x_k \underline{p_{j-k}}$. In this case, \mathbf{p} is called the **Toeplitz vector**.

Example: ETCS

Let $\lambda = ah_t/2h_x$. Then

$$u_{k,l+1} = \lambda u_{k-1,l} + u_{k,l} - \lambda u_{k+1,l}.$$

Is ETCS Toeplitz?

Is ETCS Toeplitz?

$$(P_h \mathbf{u}_{l+1})_j = u_{j,l+1} \stackrel{!}{=} \sum_k u_{k,l+1} p_{j-k}$$

$$p_{j-k} = \begin{cases} 1 & \text{if } k=j \\ 0 & \text{otherwise} \end{cases} \quad p_m = \delta_{0,m}$$

$$\rightarrow (Q_h \mathbf{u}_l)_j = \lambda u_{j-1,l} + u_{j,l} - \lambda u_{j+1,l} \stackrel{!}{=} \sum_k u_{k,l} q_{j-k}$$

$$q_{j-k} = \begin{cases} \lambda & \text{if } k=j-1 \\ 1 & \text{if } k=j \\ -\lambda & \text{if } k=j+1 \end{cases} \quad q_m = \begin{cases} \lambda & m=1 \\ 1 & \text{if } m=0 \\ -\lambda & \text{if } m=-1 \\ 0 & \text{otherwise} \end{cases}$$

$j-k=h$

Fourier Transforms of Toeplitz Operators (1/3)

$$y_j = \sum_k x_k p_{j-k}$$

$$\begin{aligned} \hat{y}(\theta) &= \sum_j \sum_k x_k p_{j-k} e^{-i\theta j} \\ &= \sum_j \sum_k \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{x}(\varphi) e^{i\varphi k} d\varphi \right) p_{j-k} e^{-i\theta j} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{x}(\varphi) \sum_j \sum_k e^{i\varphi k} p_{j-k} e^{-i\theta j} \underbrace{e^{-i\varphi j} e^{i\varphi j}}_1 d\varphi \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{x}(\varphi) \sum_j \left(\sum_k e^{i\varphi(k-j)} p_{j-k} \right) e^{i(\varphi-\theta)j} d\varphi \\ &\quad \underbrace{\hspace{10em}}_{\mathcal{L} \rightarrow \hat{p}(\varphi)^*} \end{aligned}$$

Fourier Transforms of Toeplitz Operators (2/3)

$$\hat{y}(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{x}(\varphi) \sum_j \left(\sum_k e^{i\varphi(k-j)} p_{j-k} \right) e^{i(\varphi-\theta)j} d\varphi.$$

Note: A blue bracket above the inner sum in the equation is labeled $\hat{p}(\varphi)$.

$$\sum_k e^{i\varphi(k-j)} p_{j-k} = \sum_k e^{-i\varphi(j-k)} p_{j-k} = \sum_{\ell} e^{i\varphi\ell} p_{\ell} = \hat{p}(\varphi)$$

Note: A blue arrow points from the boxed equation to the inner sum in the main equation above. A blue arrow also points from the boxed equation to the $\hat{p}(\varphi)$ label in the main equation.

Fourier Transforms of Toeplitz Operators (3/3)

$$\hat{y}(\theta) = \int_{-\pi}^{\pi} \hat{x}(\varphi) \hat{p}(\varphi) \frac{1}{2\pi} \sum_j e^{i(\varphi-\theta)j} d\varphi.$$

$$w_j = \frac{1}{2\pi} e^{i\varphi j}$$

$$\text{Then } \hat{w}(\theta) = \frac{1}{2\pi} \sum_k e^{i(\varphi-\theta)k}$$

$$\hat{y}(\theta) = \int_{-\pi}^{\pi} \hat{x}(\varphi) \hat{p}(\varphi) \hat{w}(\theta) d\varphi$$

To determine $\hat{w}(\theta)$

$$\frac{1}{2\pi} e^{i\varphi j} = w_j = \frac{1}{2\pi} \int \underbrace{\hat{w}(\theta)}_{\delta(\varphi-\theta)} e^{i\theta j} d\theta$$

$$\Rightarrow \hat{y}(\theta) = \hat{x}(\theta) \hat{p}(\theta).$$

$$u(x) \leftarrow \int u\left(\frac{z}{\epsilon}\right) \underbrace{\delta_{\epsilon}(z-x)}_{\leftarrow} dz$$



$$\int \delta_{\epsilon} = 1$$

" distribution

Fourier Transforms of Inverse Toeplitz Operators

Fourier transform $P_h^{-1} Q_h \mathbf{y}$?

$$\frac{\hat{q}(\theta)}{\hat{p}(\theta)} \mathbf{y}(\theta)$$

Bounding the Operator Norm

Bound $\|P_h^{-1}Q_h\|_2^2$ using Fourier:

.

Is the upper bound attained?