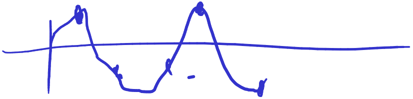
- HW 2
- Office hours moved
- Inst. message link



Dispersion / Dissipation

$z^{(x,t)} = e^{i(kx-\omega t)}$

$u_t = \cdot \sim$

$\underline{(-i\omega)}\cancel{z} = (\underline{ik + \cdot \sim})\cancel{z}$

$V^{-1}AV = D$

$A\vec{v}_1 = \lambda_1 \vec{v}_1$

$\underline{T} z = \underline{\lambda(\omega,k)} z$

Numerical D/D

$z_{j,l} = e^{i(jkh_x - l\omega h_t)}$

$P_h \vec{z}_{l+1} = Q_h \vec{z}_l$

$\hat{p}(kh_x)e^{-i\omega h_t} \vec{z}_l = \hat{q}(\underbrace{kh_x}_{\kappa}) \vec{z}_l$

$e^{-i\omega h_t} = s(\kappa)$

$\omega(\kappa)$

$\kappa = kh_x$ = waves per point

large → poorly resolved → cost

small → well-resolved

Numerical Dispersion/Dissipation

Finite difference scheme $P_h \boldsymbol{u}_{\ell+1} = Q_h \boldsymbol{u}_\ell$ with symbol $s(k)$.

$$z_{j,\ell} = z_0 e^{\log|s(\kappa)|\ell} e^{ik\left(jh_x - \frac{-\varphi(\kappa)}{kh_t}\ell h_t\right)}$$

Re $\omega(k)$: dispersion

Im $\omega(k)$: dissipation

When is the scheme dissipative?

$$|s(kh_x)| < 1$$

What is the phase speed?

$$v_{ph} = \frac{-\varphi(\kappa)}{kh_t}$$

Dispersion?

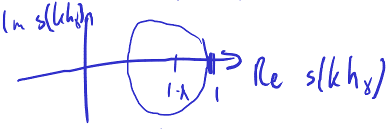
if v_{ph} does not depend k: all waves move w/ same speed

otherwise: dispersive

Dispersion/Dissipation Analysis of ETBS

Let $\lambda = ah_t/h_x$. Shown earlier: $s(kh_x) = 1 - \lambda(1 - e^{-ikh_x})$.

$$s(kh_x) = (1-\lambda) + \lambda e^{-ikh_x}$$



Dissipation $|s(k)|$ large ≤ 1

smaller $\lambda \Rightarrow$ less dissipation per step.

$\exp(x) \approx 1+x$

$$e^{-i\omega(k)h_t} = s(k) = 1 - \lambda(1 - e^{-ik})$$

Dispersion/Dissipation Analysis of ETBS: Fine Grid

$$e^{-i\omega(\kappa)h_t} = 1 - \lambda(1 - e^{-ikh_x})$$

$$S(\kappa) \approx 1 - \lambda + \lambda(1 - i\kappa) = 1 - \lambda i \kappa$$

$$e^{-i\omega(\kappa)h_t} \approx 1 - i\,\omega(\kappa)\,h_t$$

$$1 - i\,\omega(\kappa)\,h_t \simeq 1 - i\,\omega(\kappa)\,h_t$$

$$\omega(kh_x) \approx \lambda k h_x = \frac{a h_t}{\cancel{h_x}} k \cancel{h_x} = a h_t k$$

$$v_{ph} = \frac{\omega(kh_x)}{kh_t} \approx a$$

Dispersion/Dissipation: Demo

- **Demo:** Experimenting with Dispersion and Dissipation [cleared]
- **Demo:** Dispersion and Dissipation [cleared]

Outline

Introduction

Finite Difference Methods for Time-Dependent Problems
1D Advection
Stability and Convergence
Von Neumann Stability
Dispersion and Dissipation
A Glimpse of Parabolic PDEs

Finite Volume Methods for Hyperbolic Conservation Laws

Finite Element Methods for Elliptic Problems

Discontinuous Galerkin Methods for Hyperbolic Problems

Heat Equation

Heat equation ($D > 0$):

$\Rightarrow u_t - D u_{xx} = 0$
$= \delta(x$

$$\begin{aligned} u_t &= D u_{xx}, & (x,t) &\in \mathbb{R} \times (0,\infty), \\ u(x,0) &= g(x) & x &\in \mathbb{R}. \end{aligned}$$

Fundamental solution ($g(x) = \delta(x)$):

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}$$

Why is this a weird model?

∞ speed of propagation

Schemes for the Heat Equation

Cook up some schemes for the heat equation.

Explicit Euler:

Implicit Euler: