Heat Equation

- HW2
- Office hrs: 4-5 today
- Project 1

Heat equation ($D > 0$):

$$\begin{aligned} u_t &= Du_{xx}, \; = f \quad (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) &= g(x) \qquad x \in \mathbb{R}. \end{aligned}$$

Fundamental solution ($g(x) = \delta(x)$):

spreading Gaussian ; Fast Gauss transform

Why is this a weird model?

∞ speed of propagation

Schemes for the Heat Equation

$$u_{xx} \approx \frac{u(x-h) - 2u(x) + u(x+h)}{h^2}$$

Cook up some schemes for the heat equation.

Explicit Euler:

$$\boxed{\frac{u_{k,\ell+1} - u_{k,\ell}}{h_t} - D\,\frac{u_{k+1,\ell} - 2u_{k,\ell} + u_{k-1,\ell}}{h_x^2} = 0}$$

Implicit Euler:

$$\boxed{\frac{u_{k,\ell+1} - u_{k,\ell}}{h_t} - D\,\frac{u_{k+1,\ell+1} - 2u_{k,\ell+1} + u_{k-1,\ell+1}}{h_x^2} = 0}$$

Von Neumann Analysis of Explicit Euler for Heat (1/2)

Let $\lambda = Dh_t/h_x^2$.

$P_h \vec{u}^{\ell+1} = Q_h \vec{u}^{\ell}$

$$u_{k,\ell+1} = u_{k,\ell} + \lambda(u_{k+1,\ell} - 2u_{k,\ell} + u_{k-1,\ell}).$$

$P_h = I \qquad Q_h = \mathrm{tridiag}(\lambda, 1-2\lambda, \lambda)$

So

$\hat{p}(\varphi) = 1$

$\hat{q}(\varphi) = \lambda e^{-i\varphi} + (1-2\lambda) + \lambda e^{i\varphi} = 1 - 2\lambda + 2\lambda\cos(\varphi)$

Want $|\hat{s}(\varphi)| \le 1 \iff |\hat{q}(\varphi)| \le 1$

$-1 \le 1 + 2\lambda(\cos\varphi - 1) \le 1$

$-2 \le 2\lambda(\cos\varphi - 1) \le 0$

Von Neumann Analysis of Explicit Euler for Heat (2/2)

$$-2 \le 2\lambda(\cos(\varphi) - 1) \le 0.$$

$-2 \le (\cos\varphi - 1) \le 0$ if $\lambda \ge 0$, holds.

$-2 \le -4\lambda \iff \frac{1}{2} \ge \frac{D h_t}{h_x^2} \iff h_t \le \frac{h_x^2}{2D}$

Comment on the stability region found regarding speeds of propagation.

- heat eqn has ∞ speed of info prop.
- explicit scheme propagates info one cell per step

papers over the mismatch

Von Neumann Analysis of Implicit Euler for Heat

$u \cdot u$

Let $\lambda = Dh_t/h_x^2$.

$$u_{k,\ell+1} - \lambda(u_{k+1,\ell+1} - 2u_{k,\ell+1} + u_{k-1,\ell+1}) = u_{k,\ell}$$

$P_h = \operatorname{tridiag}(-\lambda, 1+2\lambda, -\lambda) \quad Q_1 = I$

$\hat{p}(\varphi) = 1 + 2\lambda(1-\cos(\varphi)) \quad \hat{q}(\varphi) = 1$

$|s(\varphi)| \le 1 \qquad 1 \le \left| 1 + 2\lambda\underbrace{(1-\cos\varphi)}_{0 \le 2} \right| \quad \checkmark$

Does the type of system we need to solve for implicit+parabolic correspond to another PDE?

$u_t = D(u_{xx} + u_{yy})$

$\dfrac{u_{\ell+1} - u_\ell}{h_t} = D(u_{xx}^{\ell+1} + u_{yy}^{\ell+1})$ → elliptic

Δu

$$u_t + a u_x = 0$$ advection

$$u_t + \underline{u} u_x = 0$$

10 $$\frac{\partial u}{\partial t} + u \cdot \nabla u + \cancel{\frac{\nabla p}{\rho}} = g$$

Euler's equations (momentum eq'n part of that)

$$u_t + (f(u))_x = 0$$

$f(u) = \frac{u^2}{2}$ $f'(u) = u$

$$u_t + \left(\frac{u^2}{2}\right)_x = 0$$ $\longleftarrow$ Burger's equation

$$u_t + f(u)_x = 0$$

$$\int_a^b dx \quad \Big(\quad \Big)$$

$$\partial_t \int_a^b u(t)\, dx + \int_a^b f(u)_x\, dx = 0$$

$$\partial_t \int_a^b u(t)\, dx + f(b) - f(a) = 0$$

Outline

Outline

Conservation Laws: Recap

$$u_t + f(u)_x = 0,$$

where u is a function of x and $t \in \mathbb{R}_0^+$.

Rewrite in integral form:

$$\partial_t \int_a^b u(t)\, dx + f(b) - f(a) = 0$$

Recall: Characteristic Curve: a function $x(t)$ so that $u(\underline{x(t)}, t) = u(x_0, 0)$.

$$\begin{cases} \frac{\mathrm{d}x(t)}{\mathrm{d}t} = f'(u(x(t), t)), \\ x(0) = x_0. \end{cases}$$

What assumption underlies all this?

Smooth solution

Burger's Equation

Consider Burgers' Equation:

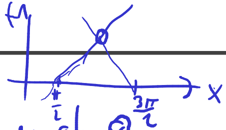
$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0, \\ u(x,0) = g(x) = \sin(x). \end{cases}$$

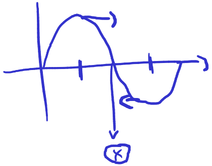
Interpret Burger's equation.

$f(u) = \frac{u^2}{2}$ $\quad f'(u) = u$

$\rightsquigarrow$ wave breaking

Consider the characteristics at $\pi/2$ and $3\pi/2$.





solution gets steeper at ⊗, discontinuous!

Weak Solutions

$\int u'v = [uv] - \int uv'$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b u(x,t)\mathrm{d}x = f(u(a,t)) - f(u(b,t)) \qquad \otimes$$

Define a weak solution:

- Option 1: If $\otimes$ ("integral form") holds for all subintervals (a,b), then we might call u some type of solution
  Observe: discontinuities allowed!
- Option 2: $\int_0^\infty \int_{-\infty}^\infty u_t \varphi + f(u)_x \varphi \, dx\, dt = 0 \quad \varphi \in C_0^1(\mathbb{R}\times[0,\infty))$

$$- \int_0^\infty \int_{-\infty}^\infty (u\varphi_t + f(u)\varphi_x)\, dx\, dt$$

$$- \int u^0(x)\varphi(x,0)\, dx = 0$$

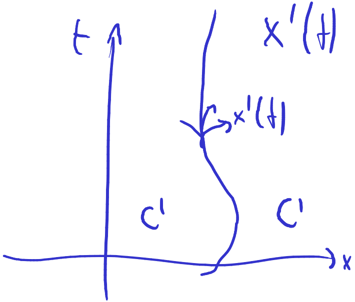
turns out:

O1 and O2 are equiv.

math notion of "weak derivative"

Rankine-Hugoniot Condition (1/2)

Consider: Two C^1 segments separated by a curve $x(t)$ with no regularity.



$$x'(t) = \frac{f(u^+) - f(u^-)}{u^+ - u^-}$$

Rankine-Hugoniot Condition (2/2)

$$(d/dt)G_a(x(t),t) = u(x(t),t)x'(t) - (f(u(x(t),t)) - f(u(a,t))).$$