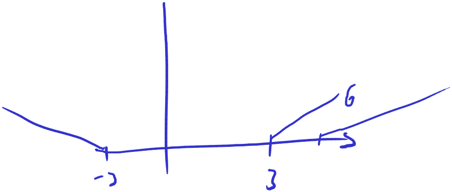
① $u_{j+1,l} \to e^{i\theta}$

$u_{j-1,l} \to e^{-i\theta}$

$$u_{j,l+1} = \frac{1}{2}\left(u_{j+1,l} + u_{j-1,l}\right) - \frac{\lambda}{2}\left(u_{j+1,l} - u_{j-1,l}\right)$$

$$\hat{G}(\theta) = e^{i\theta}\left(\frac{1}{2} - \frac{\lambda}{2}\right) + e^{-i\theta}\left(\frac{1}{2} + \frac{\lambda}{2}\right)$$



Weak Solutions

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b u(x,t)\mathrm{d}x = f(u(a,t)) - f(u(b,t))$$

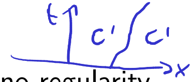
Define a weak solution:

– "Integral form of the c. law"

$$-\int_0^T\int_\Omega (u_t + f(u)_x)\,\varphi \, dx\, dt$$

→ equivalent

Rankine-Hugoniot Condition (1/2)



Consider: Two C^1 segments separated by a curve $x(t)$ with no regularity.

$$\to \frac{d}{dt}\Big(\underbrace{\int_a^{x(t)\,(z)} u(x,t)\,dx}_{G_a(z,t)\ C^1\ x(t)} + \underbrace{\int_{x(t)\,(z)}^{b} u(x,t)\,dx}_{G_b(z,t)\ C^1\ x(t)}\Big) + f(u(b,t)) - f(u(a,t)) = 0$$

$$\partial_t G_a$$

$$\frac{d}{dt} G_a(x(t),t) = \frac{\partial G_a(x(t),t)}{\partial z}\, x'(t) + \frac{\partial G_a}{\partial t}$$

$$\partial_1 G_a$$

$$= u(x,t)\, x'(t) + \int_a^{x(t)} u_t(x,t)\,dx$$

$$= u(x,t)\, x'(t) - \int_a^{x(t)} f(u(x,t))_x\,dx$$

$$= u(x(t),t)\, x'(t) - \big(f(u(x(t),t)) - f(u(a,t))\big)$$

$$f(x) = \int_a^x u(\tau)\,d\tau$$

$$f'(x) = u(x)$$

Rankine-Hugoniot Condition (2/2)

$$(d/dt)G_a(x(t),t) = u(x(t),t)x'(t) - (f(u(x(t),t)) - f(u(a,t))).$$

$$x'(t) = \frac{f(u^+) - f(u^-)}{u^+ - u^-} = \frac{[f(u)]}{[u]}$$

Rankine-Hugoniot and Weak Solutions

Theorem (Rankine-Hugoniot and Weak Solutions)

If u is piecewise C^1 and is discontinuous only along isoated curves, and if u satisfies the PDE when it is C^1, and the Rankine-Hugoniot condition holds along all discontinuous curves, then u is a weak solution of the conservation law.