Conservation of Entropy?

$\eta(u)_t + \psi(u)_x \le 0$

What can you say about conservation of entropy in time?

$$0 \ge \int_{t_1}^{t_2}\int_{x_1}^{x_2} \eta(u)_t + \psi(u)_x \, dx\, dt$$

$t_2 > t_1$

$$= \left[\int_{x_1}^{x_2} \eta(u)\, dx\right]_{t_1}^{t_2} + \left[\int_{t_1}^{t_2} \psi(u)\, dt\right]_{x_1}^{x_2}$$

in/out flow $=0$

$$\underbrace{\int_{x_1}^{x_2} \eta(u(x,t_2))\, dx}_{\text{Total entropy in } [x_1,x_2] \text{ @ } t_2} \le \underbrace{\int_{x_1}^{x_2} \eta(u(x,t_1))\, dx}_{\text{TE in } [x_1,x_2] \text{ @ } t_1} \underbrace{- \left[\int_{t_1}^{t_2} \psi(u)\, dt\right]_{x_1}^{x_2}}_{\text{In/out flow of entropy}}$$

phys entropy $= -$ math. entropy

"In a closed system phys. entropy can only increase."

$\eta \to$ math. entropy can only decrease.

Total Variation

$$\mathrm{TV}(u) = \limsup_{\varepsilon\to 0} \frac{1}{\varepsilon}\int |u(x+\varepsilon)-u(x)|\,dx.$$

Simpler form if u is differentiable?

$$\mathrm{TV}(u) = \int |u'|$$

Hiking analog?

Elevation change

$u_t + f(u)_x = 0$ $\qquad$ $u_t + \nabla\cdot f(u)$

$\to v = u_x$

Total Variation and Conservation Laws

Theorem (Total Variation is Bounded [Dafermos 2016, Thm. 6.2.6])

Let u be a solution to a conservation law with $f''(u) \geq 0$. Then:

$$\mathrm{TV}(u(t+\Delta t,\cdot)) \leq \mathrm{TV}(u(t,\cdot)) \quad \textit{for } \Delta t \geq 0.$$

- smooth solutions / non-crossing char. ⇒ TV constant
- shock ⇒ some characteristics disappear ⇒ TV decreases

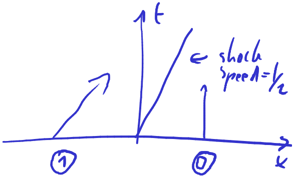
Theorem (L^1 contraction [Dafermos 2016, Thm. 6.3.2])

Let u, v be viscosity solutions of the conservation law. Then

$$\|u(t+\Delta,\cdot) - v(t+\Delta t,\cdot)\|_{L^1(\mathbb{R})} \leq \|u(t,\cdot) - v(t,\cdot)\|_{L^1(\mathbb{R})} \quad \textit{for } \Delta t \geq 0.$$

Outline

Finite Difference for Conservation Laws? (1/2)



$$\begin{cases} u_t + \left(\frac{u}{2}\right)^2_x = 0 \\ u(x,0) = \begin{cases} 1 & x < 0, \\ 0 & x \geq 0. \end{cases} \end{cases}$$

Entropy Solution?

$$u(x,t) = \begin{cases} 1 & x < \frac{1}{2}t \\ 0 & x > \frac{1}{2}t \end{cases}$$

$$s = \frac{f(u^+) - f(u^-)}{u^+ - u^-} = \frac{0 - \frac{1}{2}}{0 - 1} = \frac{1}{2}$$

Rewrite the PDE to 'match' the form of advection $u_t + au_x = 0$:

$$u_t + uu_x = 0$$

Equivalent?

Finite Difference for Conservation Laws? (2/2)

Recall the *upwind scheme* for $u_t + au_x = 0$:

$$u_{j,\ell+1} - u_{j,\ell} = a \cdot \frac{\Delta t}{\Delta x}\left(u_{j,\ell} - u_{j-1,\ell}\right)$$

Write the upwind FD scheme for $u_t + uu_x = 0$:

$$u_{j,\ell+1} - u_{j,\ell} = u_{j,\ell} \frac{\Delta t}{\Delta x}\left(u_{j,\ell} - u_{j-1,\ell}\right)$$

$$u(x,0) = \begin{cases} 1 & x < 0 \\ 0 & > 0 \end{cases}$$

$j = 0: \quad u_{0,\ell} = 0$

$j \neq 0: \quad u_{j,\ell} = u_{j,0}$

$\Rightarrow$ Does not converge to a weak solution!

Schemes in Conservation Form

Definition (Conservative Scheme)

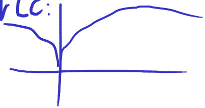
A conservation law scheme is called conservative iff it can be written as

$$u_{j,\ell+1} = u_{j,\ell} - \frac{\Delta t}{\Delta x}\left[f^*_{j+\frac{1}{2}}(\vec{u}_\ell) - f^*_{j-\frac{1}{2}}(\vec{u}_\ell)\right]$$

where $f^*\dots$

- Lipschitz continuous
- $f^*(u, u, u, \dots, u) = f(u)$

not LC:



$\sqrt{|x|}$

Theorem (Lax-Wendroff)

If the solution $\{u_{j,\ell}\}$ to a conservative scheme converges (as $\Delta t, \Delta x \to 0$) boundedly almost everywhere to a function $u(x,t)$, then u is a weak solution of the conservation law.

Lax-Wendroff Theorem: Proof

Summation by parts: With $\Delta^+ a_k = a_{k+1} - a_k$ and $\Delta^- a_k = a_k - a_{k-1}$:

$$\sum_{k=1}^{N} a_k(\Delta^- \varphi_k) + \sum_{k=1}^{N} \varphi_k(\Delta^+ a_k) = -a_1\varphi_0 + \varphi_N a_{N+1}.$$

Let $\varphi_{j,\ell} = \varphi(x_j, t_\ell)$ for $\varphi \in C_0^1$ (compact support)

$$0 = \sum_{\ell=1}^{\infty} \sum_{j} \left(\frac{\Delta^+ u_{j,\ell}}{h_t} + \frac{\Delta^+ f^*_{j-\frac{1}{2},\ell}}{h_x} \right) \varphi_{j,\ell} h_x h_t$$

Finite Volume Schemes

Finite volume: Idea?