"conservative":

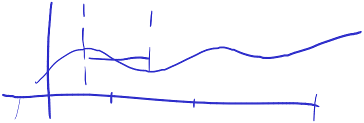
$$u_{j,\ell+1} - u_{j,\ell} + \frac{h_t}{h_x}\left(f^*_{j+1/2} - f^*_{j-1/2} \right) = 0$$

$$f^*(u, \dots, u) = f(u)$$

f^* Lipschitz

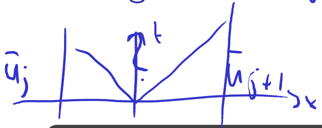
L-W thm: conservative $\Rightarrow$ converges to weak solution

FV: DOFs are cell averages $\rightarrow \bar{u}_j = \frac{1}{h_x}\int_{x_{j-1/2}}^{x_{j+1/2}} u$



Flux Integrals?

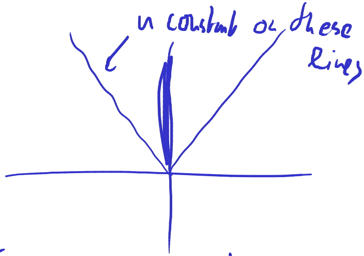
$\bar u_{j,\ell+1} - \bar u_{j,\ell} + \frac{?}{}\left(@_{j+\frac12} - @_{j-\frac12}\right) = 0$



$$\frac{1}{h_x}\int_{t_\ell}^{t_{\ell+1}} f(u_{j+1/2})\mathrm{d}t?$$

Change of vars: $\tilde x = a x$, $\tilde t = a t$

leaves IC and c.law invariant: $u_{\tilde t} + f(u)_{\tilde x} = 0$



$\Rightarrow u(x,t) = \tilde u(x/t)$

So u is constant along $x = x_{j\pm 1/2}$, so that

$$\frac{1}{h_x}\int_{t_\ell}^{t_{\ell+1}} f(u_{j+\frac12}) = \frac{h_t}{h_x} f(u_{j+\frac12})$$

The Godunov Scheme

Altogether:

$$\bar{u}_{j,\ell+1} = \bar{u}_{j,\ell} - \frac{h_t}{h_x}\left(f(u_{j+1/2,\ell}) - f(u_{j-1/2,\ell})\right).$$

Overall algorithm?

- Reconstruct $u^-_{j+\frac{1}{2}}$, $u^+_{j+\frac{1}{2}}$
- Evolve $\to$ solve Riemann problem
- Average $\to$ obtain new $\bar{u}_{j,\ell+1}$

Heuristic time step restriction?

$h_t \le h_x / \max_j |f'(\bar{u}_j)|$

$f^*(u^-_{j+\frac{1}{2},\ell}, u^+_{j+\frac{1}{2},\ell})$ f^{Lax}

"Riemann solver" "numerical

$u_{2+\frac{1}{2}} = \bar{u}_2$ reconstr. by constants

$u_{2+\frac{1}{2}}$ = look up poly that matches cell avg. eval. at $x+\frac{1}{2}$

$\bar{u}_1$ $\bar{u}_2$ $\bar{u}_3$

u_ℓ u_r

Riemann Problem

$f'(u) = a$

$$
\begin{cases} u_t + f(u)_x = 0, \\ u(x,0) = \begin{cases} u_l & x < 0, \\ u_r & x \geq 0 \end{cases} \end{cases}
$$

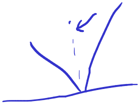
Exact solution in the Burgers case?

$$
u(x,t) = \begin{cases} \begin{cases} u_l & x < st \\ u_r & x \geq st \end{cases} & u_l \geq u_r \quad \text{"if shock"} \\[1em] \begin{cases} u_l & x < u_l t \\ x/t & \text{otherwise} \\ u_r & x \geq u_r t \end{cases} & u_l < u_r \quad \text{if rarefaction} \end{cases}
$$

$$
s = \frac{[f]}{[u]} = \frac{f(u_r) - f(u_l)}{u_r - u_l} = \frac{\frac{1}{2}\left[u_r^2 - u_l^2\right]}{u_r - u_l} = \frac{u_l + u_r}{2}
$$

Rieamnn Solver for a General Conservation Law

To complete the scheme: Need $f^*(u^-, u^+)$. For Burgers: already known.
For a general convex ($f''(u) > 0$) conservation law?



$$f^*(u(0,t)) = f^*(u^-,u^+) = \begin{cases} f(u^-) & \text{if shock with } s>0 \\ f(u^+) & \text{if shock with } s\le 0 \\ f(u^-) & \text{if rarf } f'(u^-)\ge 0 \\ f(u^+) & \text{if rarf } f'(u^+)\le 0 \\ f(u_s) & \text{if rarf } f'(u^-) < 0 < f'(u^+) \end{cases}$$

solve for stagnation state: $f'(u_s) = 0$

Equivalent to

$$f^*(u^-, u^+) = \begin{cases} \max_{u^+ \le u \le u^-} f(u) & \text{if } u^- > u^+, \\ \min_{u^- \le u \le u^+} f(u) & \text{if } u^- \le u^+. \end{cases}$$

More Riemann Solvers

Downside of Godunov Riemann solver?

Ugh ... solve $f'(u_*) = 0$?

Back to Advection

$f^*(u^-, u^+)$

Consider only $f(u) = au$ for now. Riemann solver inspiration from FD?

For $a \geq 0$, ETBS:

$$0 = \frac{u_{j,\ell+1} - u_{j,\ell}}{h_t} + a\frac{u_{j,\ell} - u_{j-1,\ell}}{h_x}$$

$$= \frac{u_{j,\ell+1} - u_{j,\ell}}{h_t} + \frac{f(u_{j,\ell}) - f(u_{j-1,\ell})}{}$$

$$= \frac{u_{j,\ell+1} - u_{j,\ell}}{h_t} + \frac{f^*(u_{j,\ell}, u_{j+1,\ell}) - f^*(u_{j-1,\ell}, u_{j,\ell})}{}$$

$$f^*(u^-, u^+) = \begin{cases} au^- & a \geq 0 \\ au^+ & a < 0 \end{cases} = \frac{au^- + au^+}{2} - \frac{|a|}{2}(u^+ - u^-)$$

Side Note: First Order Upwind, Rewritten

$$\frac{u_{j,\ell+1} - u_{j,\ell}}{h_t} + \frac{f^*(u_{j,\ell}, u_{j+1,\ell}) - f^*(u_{j-1,\ell}, u_{j,\ell})}{h_x}$$

with

$$f^*(u^-, u^+) = \frac{au^- + au^+}{2} - \frac{|a|}{2}(u^+ - u^-).$$

$$\underbrace{\frac{u_{j,\ell+1} - u_{j,\ell}}{h_t} + a\frac{u_{j+1,\ell} - u_{j-1,\ell}}{2h_x}}_{\text{ETCS}} = \underbrace{\frac{|a|h_x}{2}\frac{u_{j+1,\ell} - 2u_{j,\ell} + u_{j-1,\ell}}{h_x^2}}_{\text{Dissipation}}$$

2nd order discr. of u_{xx}!

"numerical dissipation"

Lax-Friedrichs

Generalize linear upwind flux for a nonlinear conservation law:

$$f^*(u^-, u^+) = \frac{au^- + au^+}{2} - \frac{|a|}{2}(u^+ - u^-).$$

Demo: Finite Volume Burgers [cleared] (Part I)