$$u_{j,l+1} = G(u_{j-p,l}, \ldots, u_{j+p,l})$$

'monotone' $\quad \uparrow \quad \uparrow \quad \uparrow$

LLF is ✓

Monotone Schemes: Properties

Theorem (Good properties of monotone schemes)

- *Local maximum principle:*

$$\min_{i \in \text{stencil around } j} u_i \leq G(u)_j \leq \max_{i \in \text{stencil around } j} u_i.$$

- *L^1-contraction:*

$$\|G(u) - G(v)\|_{L^1} \leq \|u - v\|_{L^1}.$$

- *TVD:*

$$TV(G(u)) \leq TV(u).$$

- *Solutions to monotone schemes satisfy all entropy conditions.*

Godunov's Theorem

Theorem (Godunov)

Monotone schemes are at most first-order accurate.

What now?

Relax the cond? Just TVD?

Linear Schemes

Definition (Linear Schemes)

A scheme is called a linear scheme if it is linear when applied to a linear PDE:

$$u_t + au_x = 0,$$

where a is a constant.

Write the general case of a linear scheme for $u_t + u_x = 0$:

$$u_{j,\ell+1} = \sum_{k=-K}^{K} c_k(\lambda)\, u_{j-k,\ell} = G(\ldots \quad)$$

monotone $\Leftrightarrow$ all $c_k(\lambda) \geq 0$

$\hookrightarrow$ linear + monotone $\Leftrightarrow$ "positive"

Linear + TVD = ?

Theorem (TVD for linear Schemes)

For linear schemes, TVD $\Rightarrow$ monotone.

What does that mean?

Linear, TVD $\Rightarrow$ first order. boo!

Now what?

Try for nonlinear schemes...?

$\sum |\Delta_+ u_{j,\ell}|$

Need to prove TVD... $TV(\vec{u}^\ell) = \sum_j |u_{j+1,\ell} - u_{j,\ell}|$

Harten's Lemma

Theorem (Harten's Lemma)

If a scheme can be written as

$$\bar{u}_{j,\ell+1} = \bar{u}_{j,\ell} + \lambda(C_{j+1/2}\Delta_+\bar{u}_j - D_{j-1/2}\Delta_-\bar{u}_j)$$

with $C_{j+1/2} \geq 0$, $D_{j+1/2} \geq 0$, $1 - \lambda(C_{j+1/2} + D_{j+1/2}) \geq 0$ *and* $\lambda = h_t/h_x$, *then it is TVD.*

As a matter of notation, we have

$$\begin{aligned} \Delta_+ u_j &= u_{j+1} - u_j, \\ \Delta_- u_j &= u_j - u_{j-1}. \end{aligned}$$

We have omitted the time subscript for the time level ℓ.

Harten's Lemma: Proof

$$\Delta_+ u_{j,\ell+1} = \Delta_+ u_{j,\ell} + \lambda \Delta_+ \left(C_{j+\frac{1}{2}} \Delta_+ u_j - D_{j-\frac{1}{2}} \Delta_- u_j \right)$$

$$= \Delta_+ u_j + \lambda \Big(C_{j+\frac{3}{2}} \Delta_+ u_{j+1} - D_{j+\frac{1}{2}} \cancel{\Delta_- u_{j+1}}\ \Delta_+ u_j$$

$$- C_{j+\frac{1}{2}} \Delta_+ u_j + D_{j-\frac{1}{2}} \Delta_- u_j \Big)$$

$$= \left(1 - \lambda \left(C_{j+\frac{1}{2}} + D_{j+\frac{1}{2}} \right) \right) \Delta_+ u_j$$

$$+ \lambda C_{j+\frac{3}{2}} \Delta_+ u_{j+1} + \lambda D_{j-\frac{1}{2}} \Delta_- u_j$$

$$|\Delta_+ u_{j,\ell+1}| \le \underset{\varepsilon_j}{\left(1 - \lambda \left(C_{j+\frac{1}{2}} + D_{j+\frac{1}{2}} \right) \right)} |\Delta_+ u_j|$$

$$+ \underbrace{\lambda C_{j+\frac{3}{2}} |\Delta_+ u_{j+1}|}_{\varepsilon_{j'}\, C_{j'+\frac{1}{2}} |\Delta_+ u_{j'}|} + \underbrace{\lambda D_{j-\frac{1}{2}} |\Delta_- u_j|}_{\varepsilon_{j''}\, D_{j''+\frac{1}{2}} |\Delta_+ u_{j''}|}$$

$$TV(\vec{u}^{\ell+1}) = \sum |\Delta_+ u_{j,\ell+1}| \le \sum_j \left(1 - \lambda \left(C_{j+\frac{1}{2}} + D_{j+1/2} \right) + \lambda C_{j+\frac{1}{2}} + \lambda D_{j+\frac{1}{2}} \right) |\Delta_+ u_j|$$

$$= TV(\vec{u}^{\ell})$$

Minmod Scheme

Still assume $f'(u) \geq 0$.

$$f^{*,(1)}_{j+1/2} = f\Big(\bar{u}_j + \underbrace{\frac{1}{2}(\bar{u}_{j+1} - \bar{u}_j)}_{\tilde{u}_j^{(1)}}\Big), \qquad f^{*,(2)}_{j+1/2} = f\Big(\bar{u}_j + \underbrace{\frac{1}{2}(\bar{u}_j - \bar{u}_{j-1})}_{\tilde{u}_j^{(2)}}\Big).$$

central

upwind

Design a 'safe' thing to use for $\tilde{u}$:

$$\operatorname{minmod}(a,b) = \begin{cases} a & |a| < |b|,\ ab > 0 \\ b & |b| \le |a|,\ ab > 0 \\ 0 & ab \le 0 \end{cases}$$

$$\hat{u}_j = \operatorname{minmod}\left(\tilde{u}_j^{(1)}, \tilde{u}_j^{(2)}\right)$$

slopes

"slope limiter"

$$f^{*,(3)}_{j+\frac12} = f(\bar{u}_j + \hat{u}_j)$$

@ local extremum which drives TVD growth

Minmod is TVD

Show that Minmod is TVD: