## Back to the Model Problem

$$-\Delta u + u = 0$$

$$\begin{aligned}
a(u,v) &= \langle \nabla u, \nabla v\rangle_{L^2} + \langle u, v\rangle_{L^2} = (u,v)_{H^1} \\
g(v) &= \langle f, v\rangle_{L^2} \\
(u,v)_{H^1} = a(u,v) &= g(v) \quad \text{for all } v \in H^1_0
\end{aligned}$$

$g(\alpha v) = \alpha g(v)$

$|g(v)| \leq C \cdot \|v\|$

Have we learned anything about the solvability of this problem?

Show: $g$ is bounded as a functional on $H^1$

RRT $\Rightarrow$ There exists a $u \in H^1_0$ s.t. $g(v) = (u,v)_{H^1_0} = a(u,v)$

$\Rightarrow$ Existence and uniqueness of $u$.

# Poisson

Let $\Omega \subset \mathbb{R}^n$ open, bounded, $f \in H^{-1}(\Omega)$.

$$-\Delta u = -\nabla \cdot \nabla u = f(x)$$
$$u(x) = 0 \qquad (x \in \partial\Omega)$$

This is called the Poisson problem (with Dirichlet BCs).

Weak form?

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \qquad \forall v \in H_0^1$$

# Ellipticity

Let $V$ be Hilbert space.

## $V$-Ellipticity

A bilinear form $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ is called coercive if there exists a constant $c_0 > 0$ so that

$$c_0 \|u\|_V^2 \leq a(u,u) \qquad \text{for all } u \in V$$

and $a$ is called continuous if there exists a constant $c_1 > 0$ so that

$$a(u,v) \leq c_1 \|u\|_V \|v\|_V$$

If $a$ is both coercive and continuous on $V$, then $a$ is said to be $V$-elliptic.

# Lax-Milgram Theorem

Let $V$ be Hilbert space with inner product $\langle \cdot, \cdot \rangle$.

## Lax-Milgram, Symmetric Case

Let $a$ be a $V$-elliptic bilinear form that is also symmetric, and let $g$ be a bounded linear functional on $V$.
Then there exists a unique $u \in V$ so that $a(u, v) = g(v)$ for all $v \in V$.

$(u,v)_a = a(u,v)$. sym ✓ linearity ✓

- Need $a(u,u) \geq 0$

$a(u,u) \geq c_0 \|u\|_V \geq 0$

- Need $a(u,u) = 0 \Rightarrow u = 0$.

RRT $0 = a(u,u) \geq c_0 \|u\|_V^2 \geq 0 \Rightarrow u = 0$.

$\Rightarrow$ existence and uniqueness.

# Back to Poisson

$\|u\|_{H^1} = \|u\|_{L^2} + \|\nabla u\|_{L^2}$

Can we declare victory for Poisson?

stab: $\left|\int \nabla u \cdot \nabla v\right| \le \|\nabla u\|_{L^2} \|\nabla v\|_{L^2} \le c_1 \|u\|_{H^1} \|v\|_{H^1}$ ✓

coercive: $\int_\Omega \nabla u \cdot \nabla u \ge c_1 \left( \int \nabla u \cdot \nabla u \, dx + \int_\Omega u^2 \, dx \right)$

Can this inequality hold in general, without further assumptions?

constants break coercivity ☹

# Poincaré-Friedrichs Inequality (1/3)

**Theorem (Poincaré-Friedrichs Inequality)**

*Suppose $\Omega \subset \mathbb{R}^n$ is bounded and $u \in H_0^1(\Omega)$. Then there exists a constant $C > 0$ such that*

$$\|u\|_{L^2} \leq C \|\nabla u\|_{L^2} .$$

$$\nabla \cdot (u^2 \vec{x}) = \partial_1 (u^2 x_1) + \cdots + \partial_n (u^2 x_n)$$

$$= u^2 + 2(u \partial_1 u) x_1 + \cdots + u^2 + 2(u \partial_n u) x_n$$

$$= n u^2 + 2u (\nabla u \cdot \vec{x})$$

$$u^2 = \frac{1}{n} \nabla \cdot (u^2 \vec{x}) - \frac{2}{n} (\nabla u \cdot \vec{x}) u$$

$\|u\|^2$

$\leq c \|\nabla u\| \|u\|$

# Poincaré-Friedrichs Inequality (2/3)

Prove the result in $C_0^\infty(\Omega)$.

$$\|u\|_{L^2}^2 = \int_\Omega u^2 = \int_\Omega \frac{1}{n}\nabla\cdot\left(u^2\vec{x}\right) - \frac{2}{n}\left(\nabla u\cdot\vec{x}\right)u\,dx$$

$$= \underbrace{\frac{1}{n}\int_{\partial\Omega}\hat{n}\cdot\left(u^2\vec{x}\right)dS_x}_{=0\ (H_0^1)} - \frac{2}{n}\int u\cdot\left(\nabla u\cdot\vec{x}\right)dx$$

$$\le \frac{2}{n}\max_{x\in\Omega}|\vec{x}|\int|u\nabla u|\,dx = \frac{2}{n}\max_{x\in\Omega}\|x\|\,\|u\|_{L^2}\|\nabla u\|_{L^2}$$

$$\Rightarrow \|u\|_{L^2} \le C\,\|\nabla u\|_{L^2}.$$

# Poincaré-Friedrichs Inequality (3/3)

Prove the result in $H_0^1(\Omega)$.

Inequality continues to hold in the limit because $u_n \to u$ in $H_1$

$$\Rightarrow \left.\begin{array}{l} \|u_k - u\|_{L^2} \to 0 \\ \|\nabla u_k - \nabla u\|_{L^2} \to 0 \end{array}\right\} H^1\text{-norm.}$$

# Back to Poisson, Again

Show that the Poisson bilinear form is coercive.

$$\frac{1}{C^2+1}\|u\|_{H^1}^2 = \frac{1}{C^2+1}\left(\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\right) \overset{PF}{\leq} \|\nabla u\|_{L^2} = a(u,u)$$

Draw a conclusion on Poisson:

Poisson $a(u,v) = \int \nabla u \cdot \nabla v$ is coercive and continuous

$\underset{\text{Lax-Milgram}}{\Rightarrow}$ existence and uniqueness

# Outline

Ritz-Galerkin → use same space for $u_h$, $v_h$

Some key goals for this section:

- How do we use the weak form to compute an approximate solution?
- What can we know about the accuracy of the approximate solution?

Can we pick one underlying principle for the construction of the approximation?

Let $H$ be full Hilbert space ($H^1_0$ or so)

$a(u,v) = g(v) \quad \forall v \in V$ Let $V \subseteq H$

$\Rightarrow$ Existence and uniqueness in $V$ as long as $V$ is itself a Hilbert space

Pick a finite-dim subspace $V_h$ (often poly)

$a(u_h, v_h) = g(v_h) \quad (v_h \in V_h)$

Each test function $v_h$ gives rise to a row of a matrix.

# Galerkin Orthogonality

$a(\alpha u + \beta w, v) =$
$\alpha a(u, v) + \beta a(w, v)$

$$a(u, v) = g(v) \quad \text{for all } v \in V, \quad a(u_h, v_h) = g(v_h) \quad \text{for all } v_h \in V_h.$$

Observations?

$a(u, v_h) = g(v_h)$

$-\quad a(u_h, v_h) = g(v_h)$

$a(\underbrace{u - u_h}_{\text{error!}}, v_h) = 0$

# Céa's Lemma

Let $V \subset H$ be a closed subspace of a Hilbert space $H$.

## Céa's Lemma

Let $a(\cdot, \cdot)$ be a coercive and continuous bilinear form on $V$. In addition, for a bounded linear functional $g$ on $V$, let $u \in V$ satisfy

$$a(u, v) = g(v) \qquad \text{for all } v \in V.$$

Consider the finite-dimensional subspace $V_h \subset V$ and $u_h \in V_h$ that satisfies

$$a(u_h, v_h) = g(v_h) \qquad \text{for all } v_h \in V_h.$$

Then

$$\boxed{\|u - u_h\|_V \leq \frac{c_1}{c_0} \inf_{v_h \in V_h} \|u - v_h\|_V}$$

# Céa's Lemma: Proof

Recall Galerkin orthgonality: $a(u_h - u, v_h) = 0$ for all $v_h \in V_h$. Show the result.

$$
\begin{aligned}
c_0 \|u - u_h\|^2 &\overset{\text{coerc.}}{\leq} a(u-u_h, u-u_h) \\
&= a(u-u_h, u-v_h) + \cancel{a(u-u_h, \underbrace{v_h-u_h}_{\in V_h})} \quad \text{Galerkin orth} \\
&= a(u-u_h, u-v_h) \\
&\leq c_1 \|u-u_h\|_V \|u-v_h\|_V
\end{aligned}
$$

# Elliptic Regularity

## Definition ($H^s$ Regularity)

Let $m \geq 1$, $H_0^m(\Omega) \subseteq V \subseteq H^m(\Omega)$ and $a(\cdot,\cdot)$ a $V$-elliptic bilinear form. The bilinear form $a(u,v) = \langle f, v \rangle$ for all $v \in V$ is called $H^s$ regular, if for every $f \in H^{s-2m}$ there exists a solution $u \in H^s(\Omega)$ and we have with a constant $C(\Omega, a, s)$,

## Theorem (Elliptic Regularity (cf. Braess Thm. 7.2))

*Let $a$ be a $H_0^1$-elliptic bilinear form with sufficiently smooth coefficient functions.*