$|a(u,v)| \le \|u\|_2 \|v\|_2$? ← not quite

$|a(u,v)| \le C_0 \|u\|_{H_1} \|v\|_{H_1}$ continuity

$C_1 \|u\|_{H_1}^2 \le |a(u,u)|$ coercivity

symm. Find v: $a(u,v) = 0 \ \forall v \Rightarrow \exists$ null space

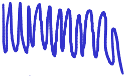
cols rows

$a(u,v) = \int \nabla u \cdot \nabla v$

$\int u' v'$

$a: H_0^1 \times H_0^1 \to \mathbb{R}$

$a: V_h \times V_h \to \mathbb{R} \to \exists!$ carries over from ∞ D.



$\sin(kx)$

Céa's Lemma

Let $V \subset H$ be a closed subspace of a Hilbert space H.

Céa's Lemma

Let $a(\cdot, \cdot)$ be a coercive and continuous bilinear form on V. In addition, for a bounded linear functional g on V, let $u \in V$ satisfy

$$a(u, v) = g(v) \qquad \text{for all } v \in V.$$

Consider the finite-dimensional subspace $V_h \subset V$ and $u_h \in V_h$ that satisfies

$$a(u_h, v_h) = g(v_h) \qquad \text{for all } v_h \in V_h.$$

Then

$\|u - u_h\|$

$$\|u-u_h\|_V \le \frac{c_1}{c_0} \inf_{v_h \in V_h} \|u - v_h\|_V$$

Céa's Lemma: Proof

Galerkin orth: $u - u_h \perp_a V_h$

Recall Galerkin orthgonality: $a(u_h - u, v_h) = 0$ for all $v_h \in V_h$. Show the result.

For any $v_h \in V_h$:

$$
\begin{aligned}
c_0 \|u-u_h\|_V^2 &\le a(u-u_h, u-u_h) \\
&= a(u-u_h, u-v_h) + \underbrace{a(u-u_h, \overbrace{v_h-u_h}^{\in V_h})}_{=0\ (GO)} \\
&\le c_1 \cancel{\|u-u_h\|}\, \|u-v_h\|
\end{aligned}
$$

since v_h was arbitrary: $\square$

Elliptic Regularity

Definition (H^s Regularity)

Let $m \geq 1$, $H_0^m(\Omega) \subseteq V \subseteq H^m(\Omega)$ and $a(\cdot,\cdot)$ a V-elliptic bilinear form. The bilinear form $a(u,v) = \langle f, v \rangle$ for all $v \in V$ is called H^s regular, if for every $f \in H^{s-2m}$ there exists a solution $u \in H^s(\Omega)$ and we have with a constant $C(\Omega, a, s)$,

$$\|u\|_{H^s} \leq C(\Omega, a, s) \|f\|_{H^{s-2m}}$$

Theorem (Elliptic Regularity (cf. Braess Thm. 7.2))

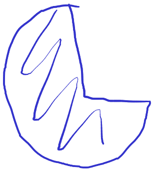
Let a be a H_0^1-elliptic bilinear form with sufficiently smooth coefficient functions.

- If Ω is convex, then the Dirichlet problem is H^2 reg.
- Let $s \geq 2$. If $\partial\Omega$ is C^s, then the Dirichlet problem is H^s reg.

Elliptic Regularity: Counterexamples

on the circle

Are the conditions on the boundary essential for elliptic regularity?

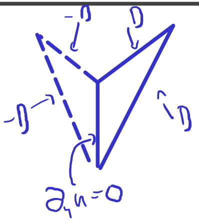


$\Delta u = 0$ $\quad u(e^{i\varphi}) = \sin(\tfrac{2}{3}\varphi)$, $u = 0$ elsewhere

$u(x,y) = \operatorname{Im}\left((x+iy)^{2/3}\right)$

Derivative: $(2/3)\, z^{-1/3} \Rightarrow u \notin H^2$!

Are there any particular concerns for mixed boundary conditions?



By symmetry, problem on whole domain (w/ reflected D) is equiv to mixed BC.

Whole-domain prob has a reentrant corner ⇒ see above.

Estimating the Error in the Energy Norm

$I_h: H_1 \to V_h$? can we? Need pt. eval.

Come up with an idea of a bound on $\|u - u_h\|_{H^1}$.

$\|u-u_h\| \leq \|u - u_h\|_{H^1} \overset{\text{Céa}}{\leq} C \cdot \inf_{v_h \in V_h} \|u - v_h\|_{H_1} \leq C \|u - I_h u\|_{H_1}$

$\overset{?}{\leq} C \cdot h^{?} \|u\|_{H^2} \leq C h \|f\|_{L^2}$ ← H^2 reg.

C: "generic constant"

Intp. bound: $\|f - p_1\|_\infty \leq C h^2 \|f''\|_\infty$

What's still to do?

- V_h?
- I_h: existence? bound?
- H^1 norm estimate? bleh.

Want L^2.

L^2 Estimates

Let H be a Hilbert space with the norm $\|\cdot\|_H$ and the inner product $\langle\cdot,\cdot\rangle$. (Think: $H = L^2$, $V = H^1$.)

Theorem (Aubin-Nitsche)

Let $V \subseteq H$ be a subspace that becomes a Hilbert space under the norm $\|\cdot\|_V$. Let the embedding $V \to H$ be continuous. Then we have for the finite element solution $u \in V_h \subset V$:

$$\|u-u_h\|_H \le C_1 \underbrace{\|u-u_h\|_V}_{} \sup_{g\in H} \left[\frac{1}{\|g\|_H} \inf_{v_h\in V_h} \|\varphi_g - v_h\|_V \right]$$

if with every $g \in H$ we associate the unique (weak) solution φ_g of the equation (also called the dual problem*)*

$a(u,v) = g(v)$

$$a(\underset{\text{test f.}}{w}, \underset{\text{sol}}{\varphi_g}) = (g, w)$$

Aubin-Nitsche: Proof

$$\|w\|_H = \sup_{g \in H} \frac{1}{\|g\|} (g, w)$$

L^2 Estimates using Aubin-Nitsche

$$\|u - u_h\|_H \leq c_1 \|u - u_h\|_V \sup_{g \in H} \left[\frac{1}{\|g\|_H} \inf_{v_h \in V_h} \|\varphi_g - v_h\|_V \right],$$

If $u \in H_0^1(\Omega)$, what do we get from Aubin-Nitsche?

So does Aubin-Nitsche give us an L^2 estimate?