Estimating the Error in the Energy Norm

$$\|u - I_h u\|_\infty \le C \cdot h \|u'\|_\infty$$
$$\le C \cdot h^2 \|u''\|_\infty$$

Come up with an idea of a bound on $\|u - u_h\|_{H^1}$.

$$\to \|(u - I_h u)'\|_\infty \le C \cdot h \|u''\|_\infty$$

$\|u - u_h\|_{L^2} \le$

$$\|u - u_h\|_{H^1} \le C \inf_{v_h \in V_h} \|u - v_h\|_{H^1} \le C \|u - I_h u\|_{H^1} \le C_1 h \|u\|_{H^2}$$

$$\underset{H^2 \text{ reg.}}{\le} C h \|f\|_{L^2}$$

What's still to do?

L^2 Estimates

Let H be a Hilbert space with the norm $\|\cdot\|_H$ and the inner product $\langle \cdot, \cdot \rangle$.
(Think: $H = L^2$, $V = H^1$.)

Theorem (Aubin-Nitsche)

Let $V \subseteq H$ be a subspace that becomes a Hilbert space under the norm $\|\cdot\|_V$. Let the embedding $V \to H$ be continuous. Then we have for the finite element solution $u \in V_h \subset V$:

$$\|u-u_h\|_H \le c_1 \underline{\|u-u_h\|}_V \cdot \sup_{g \in H\setminus\{0\}} \left[\frac{1}{\|g\|_H} \inf_{v_h \in V_h} \|\varphi_g - v_h\|_V \right]$$

if with every $g \in H$ we associate the unique (weak) solution φ_g of the equation (also called the dual problem*)*

primal $\to$ $a(u, v) = g(v)$ $\forall v$

dual $\to$ $a(w, \varphi_g) = (g, w)$ $\forall w$ $\rightsquigarrow$ corr. back to BVP?

Aubin-Nitsche: Proof

$$\|w\|_H = \sup_{g \in H \setminus \{0\}} \frac{(g,w)}{\|g\|_H} \quad \text{use for } w = u - u_h$$

$$(g, u-u_h) \overset{\text{dual}}{=} a(u-u_h, \varphi_g) \overset{\text{GO}}{=} a(u-u_h, \varphi_g - v_h)$$

$$\leq c_1 \, \|u-u_h\|_V \, \|\varphi_g - v_h\|_V$$

$$(g, u-u_h) \leq c_1 \, \|u-u_h\|_V \inf_{v_h \in V_h} \|\varphi_g - v_h\|_V$$

$$\|u-u_h\|_H = \sup_{g \in H} \frac{(g, u-u_h)}{\|g\|} \leq c_1 \|u-u_h\|_V \cdot \sup_{g \in H} \left(\frac{1}{\|g\|_H} \inf_{v_h \in V_h} \|\varphi_g - v_h\|_V \right)$$

L^2 Estimates using Aubin-Nitsche

$$\|u - u_h\|_H \leq c_1 \|u - u_h\|_V \sup_{g \in H} \left[\frac{1}{\|g\|_H} \inf_{v_h \in V_h} \|\varphi_g - v_h\|_V \right],$$

If $u \in H_0^1(\Omega)$, what do we get from Aubin-Nitsche?

For Poisson in H_0^1, dual = primal.

$$\inf_{v_h \in V_h} \|\varphi_g - v_h\|_{H^1} \leq C \|\varphi_g - I_h \varphi_g\|_{H^1} \leq C\, h \|\varphi_g\|_{H^2} \leq C \cdot h \|g\|_{L^2}$$

$$\|u - u_h\|_H \leq c_1 \|u - u_h\|_V C \cdot h \leq C \cdot h^2 \|f\|_{L^2}$$

So does Aubin-Nitsche give us an L^2 estimate?

Yes.

Outline

Finite Elements in 1D: Discrete Form

$\Omega := [\alpha, \beta]$. Look for $u \in H_0^1(\Omega)$, so that $a(u, \varphi) = \langle f, \varphi \rangle$ for all $\varphi \in H_0^1(\Omega)$. Choose $V_h = \operatorname{span}\{\varphi_1, \ldots, \varphi_n\}$ and expand $u_h = \sum_{i=1}^n u_h^i \varphi_i \in V_h$. Find the discrete system.

$$a(u_h, v_h) = (f, v_h) \quad \forall v_h$$

$$a\left(\sum_{i=1}^n u_h^i \varphi_i, v_h\right) = (f, v_h) \quad \forall v_h$$

Grids and Hats

Let $I_i := [\alpha_i, \beta_i]$, so that $\bar{\Omega} = \bigcup_{i=0}^{N} I_i$ and $I_i^\circ \cap I_j = \emptyset$ for $i \neq j$. Consider a grid

$$\alpha = x_0 < \cdots < x_N < x_{N+1} = \beta,$$

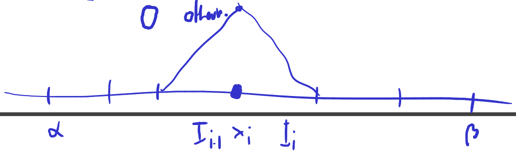
i.e. $\alpha_i = x_i$, $\beta_i = x_{i+1}$ for $i \in \{0, \ldots, N\}$. The $\{x_i\}$ are called nodes of the grid. $h_i := x_{i+1} - x_i$ for $i \in \{0, \ldots, N\}$ and $h := \max_i h_i$. V_h? Basis?

$$P_h^1 = \{ v_h \in C^0(\bar{\Omega}) : v_h|_{I_i} \in \mathbb{P}^1 \}$$

For $i \in \{0, N+1\}$:

$$\varphi_i(x) = \begin{cases} \frac{1}{h_{i-1}}(x - x_{i-1}) & x \in I_{i-1} \\ \frac{1}{h_i}(x_{i+1} - x) & x \in I_i \\ 0 & \text{otherw.} \end{cases}$$

span $\{\varphi_i\}_i = P_h^1$



Degrees of Freedom and Matrices

Define something more general than basis coefficients to solve for.

$\gamma_i : C(\bar{\Omega}) \to \mathbb{R}$ $\quad v \mapsto v(x_i) \in \mathbb{R}$

global degrees of freedom

$\operatorname{span}\{\gamma_i\} = (P_h^1)'$

dual space of

Define shape functions and assemble the stiffness matrix:

Shape functions $\hat{\varphi}_i$: $\gamma_i(\hat{\varphi}_j) = \delta_{i,j}$ for $i,j \in \{0,\dots,N+1\}$

$a(u_h, v_h) = (f, v_h)$

$(f, \hat{\varphi}_i) = a(u_h, \hat{\varphi}_i) = \sum_{j=1}^{N} \underbrace{\gamma_j(u_h)}_{\text{unknowns}} \underbrace{a(\hat{\varphi}_j, \hat{\varphi}_i)}_{\text{matrix}}$ $\quad (i = 1 \dots N)$

A Matrix Property for Efficiency

"stiffness matrix"

$a(u,v) = \int u' v'$

$a_{i,j}$

$$(A_h)_{i,j} = a(\hat{\varphi}_j, \hat{\varphi}_i).$$

Anything special about the matrix?

$\boxed{\text{tridiagonal here, sparse in general}}$

$a_{i,i} \neq 0$

$a_{i,i+1}$

$a_{i,i-1}$

otherwise $= 0$

Error Estimation

According to Céa, what's our main missing piece in error estimation now?

$$I_h^1 : C^0(\bar{\Omega}) \to P_h^1$$

$$v \longmapsto \sum_{i=0}^{N+1} \gamma_i(v)\,\varphi_i^1 \in P_h^1$$

Interpolation Error (1D-only)

For $v \in H^2(\Omega)$,

$$\| v - I_h^1 v \|_{L^2} \leq h^2 \, |v|_{H^2}$$

$$\| v - I_h^1 v \|_{H^1} \leq h \, |v|_{H^2}$$

If $v \in H^1(\Omega) \setminus H^2(\Omega)$,

$$\| v - I_h^1 v \|_{L^2} \leq h \, |v|_{H^1}$$

$$\| v - I_h^1 v \|_{H^1} \longrightarrow 0 \qquad (h \to 0)$$

Is I_h^1 defined for $v \in H^2$? for $v \in H^1 \setminus H^2$?

For general dim n, answer depends on shape of Ω.

$\rightarrow$ "Sobolev embedding theorem"

Interpolation Error: Towards an Estimate

$$a(u_h, v_h) = (f, v_h)$$

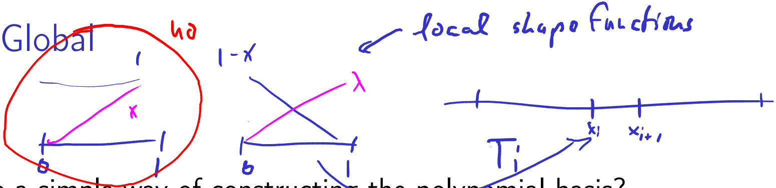
Provide an a-priori estimate.

$$\|u - u_h\|_{H^1} \le \frac{c_1}{c_0} \inf_{v_h \in P_h^1} \|u - v_h\|_{H^1} \le \frac{c_1}{c_0} \|u - I_h^1 u\|_{H^1} \le \frac{c_1}{c_0} |u|_{H^2}$$

What's the relationship between $I_h^1 u$ and u_h?

Not the same!

Local-to-Global



Is there a simple way of constructing the polynomial basis?

A FEM basis can be thought of as a composition:

- reference / local basis
- local-to-global mapping T
  reference

Local-to-Global: Math

reference coords

Construct a polynomial basis using this approach.

$$\hat{\varphi}_0(\hat{x}) = 1 - \hat{x}$$

$$\hat{\varphi}_1(\hat{x}) = \hat{x}$$

To get a basis of $P^1(I_1)$:

$$\varphi_i(x) = \begin{cases} \hat{\varphi}_1 \circ T_{i-1}^{-1}(x) & x \in [x_{i-1}, x_i] \\ \hat{\varphi}_0 \circ T_i^{-1}(x) & x \in [x_i, x_{i+1}] \end{cases}$$

global coords

Demo

Demo: Developing FEM in 1D [cleared]