Announcements

- In person next week (hopefully?)
  - 2025 C1F
- Discussion invites
- HV1
Examples: Order, Linearity?

\[(xu^2)u_{xx} + (u_x + y)u_{yy} + u_x^3 + yu_y = f\]

\[\text{quasilinear, 2nd order}\]

\[(x + y + z)u_x + (z^2)u_y + (\sin x)u_z = f\]

\[\text{semilinear, 1st order}\]
Properties of Domains

\[ u(t, x) \in C^2(\mathbb{R}^d) \quad t \in [0, T] \]

- Domain impact on existence of solution?
  - comes
  - reentrant comes
  - caps
Function Spaces: Examples

Name some function spaces with their norms.

\( C^{0}(\mathbb{R}) : \) continuous

\( C^{k}(\mathbb{R}) : k \)-times continuously differentiable

\( C^{0,1}(\mathbb{R}) : \) \[ \| f \|_{0,1} = \| f \|_{\infty} + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|} \]

\( C_{L}(\mathbb{R}) : \) "Lip sol of cond" \[ |f(x) - f(y)| \leq C \| x - y \| \]

\( L^{2}(\mathbb{R}) = \{ u : \mathbb{R} \rightarrow \mathbb{R} : \int_{-\infty}^{\infty} |u(x)|^2 \, dx < \infty \} \quad \| u \|_{2} = \sqrt{\int_{-\infty}^{\infty} |u(x)|^2 \, dx} \)

\( H^{1}(\mathbb{R}) = \{ u \in L^{2}(\mathbb{R}) : \int_{-\infty}^{\infty} |u(x)|^2 \, dx < \infty \} \quad \| u \|_{1} = \| u \|_{2} + \| u' \|_{2} \)

May also influence existence/uniqueness of solutions!
Solving PDEs

Closed-form solutions:

- If separation of variables applies to the domain: good luck with your ODE
- If not: Good luck! \( \rightarrow \) Numerics

General Idea (that we will follow some of the time)

- Pick \( V_h \subseteq V \) finite-dimensional
  - \( h \) is often a mesh spacing
- Approximate \( u \) through \( u_h \in V_h \)
- Show: \( u_h \rightarrow u \) (in some sense) as \( h \rightarrow 0 \)

Example

\( u(x) = \sin(x) \)
About grand big unifying theories

Is there a grand big unifying theory of PDEs?

NO
Collect some stamps

\[ a(x, y)u_{xx} + 2b(x, y)u_{xy} + c(x, y)u_{yy} + d(x, y)u_x + e(x, y)u_y + f(x, y)u = g(x, y) \]

<table>
<thead>
<tr>
<th>Discriminant value</th>
<th>Kind</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b^2 - ac &lt; 0 )</td>
<td>Elliptic</td>
<td>Laplace ( u_{xx} + u_{yy} = 0 )</td>
</tr>
<tr>
<td>( b^2 - ac = 0 )</td>
<td>Parabolic</td>
<td>Heat ( u_t = u_{xx} )</td>
</tr>
<tr>
<td>( b^2 - ac &gt; 0 )</td>
<td>Hyperbolic</td>
<td>Wave ( u_{tt} = u_{xx} )</td>
</tr>
</tbody>
</table>

Where do these names come from?

Search for characteristic curves

(See lecture notes by Hogg)
PDE Classification in Other Cases

Scalar first order PDEs?
- Hyperbolic

First order systems of PDEs?
- All types (ell, par, hyper) are possible, see Hogg for classification
Classification in higher dimensions

\[ Lu := \sum_{i=1}^{d} \sum_{j=1}^{d} a_{i,j}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \text{lower order terms} \]

Consider the matrix \( A(x) = (a_{ij}(x))_{i,j} \). May assume \( A \) symmetric. Why?

Schwarz's theorem

What cases can arise for the eigenvalues?

- \( \exists j : \lambda_j = 0 \) (parabolic case)
- \( \lambda_j \) all have same sign (elliptic case)
- all but one have same sign (hyperbolic case)
- more than one with different signs (ultra-hyperbolic)
Elliptic PDE: Laplace/Poisson Equation

\[ \Delta u = \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2} = \nabla \cdot \nabla u(x) \overset{2D}{=} u_{xx} + u_{yy} = f(x) \quad (x \in \Omega) \]

Called Laplace equation if \( f = 0 \). With Dirichlet boundary condition

\[ u(x) = g(x) \quad (x \in \partial \Omega). \]

Demo: Elliptic PDE Illustrating the Maximum Principle [cleared]
Elliptic PDEs: Singular Solution

**Demo:** Elliptic PDE Radially Symmetric Singular Solution [cleared]

Given $G(x) = C \log(|x|)$ as the free-space Green’s function, can we construct the solution to the PDE with a more general $f$?

\[
\Delta u = f
\]  
\[
\Rightarrow \Delta u = f
\]

What can we learn from this?

Solution is globally coupled
Elliptic PDEs: Justifying the Singular Solution

\[ u(x) = (G \ast f)(x) = \int_{\mathbb{R}^d} G(x - y)f(y)dy \]

Why?

\[ \Delta u(x) = \Delta \int_{\mathbb{R}^d} G(x - y)f(y)dy \]
\[ = \int_{\mathbb{R}^d} \Delta G(x - y)f(y)dy \]
\[ = \int_{\mathbb{R}^d} \delta(x - y)f(y)dy = f(x) \]
Parabolic PDE: Heat Equation • Separation of Variables

\[ u_t = u_{xx} \quad ((x, t) \in [0, 1] \times [0, T]) \]

\[ u(x, 0) = g(x) \quad (x \in [0, 1]) \]

\[ u(0, t) = u(1, t) = 0 \quad (t \in [0, T]) \]

Cap. \( u_{xx} + u_{yy} = 0 \)

Wave \( u_{tt} = u_{xx} \)

Plug into PDE:

\[ v(t) \cdot w(x) = \frac{v''(t)}{v(t)} = C = \frac{w''(x)}{w(x)} \]

\[ v'(t) - C \cdot v(t) \quad w''(x) = C \cdot w(x) \]

\[ v(t) = \exp(-m^2 \pi^2 t) \quad w(x) = \alpha \cdot \sin(m \pi x) \]

\[ \Rightarrow C = -m^2 \pi^2 \]
Demo: Parabolic PDE [cleared] What can we learn from analytic and numerical solution?

- "washes out" the solution
- final solution $f(x,t)$ becomes smoother
Hyperbolic PDE: Wave Equation

\[ u_{tt} = c^2 u_{xx} \quad ((x, t) \in \mathbb{R} \times [0, T]) \]
\[ u(x, 0) = g(x) \quad (x \in \mathbb{R}) \]

with \( g(x) = \sin(\pi x) \).

Is this problem well-posed?

Can be rewritten in conservation law form: