Numerical Differentiation: How?

How can we take derivatives numerically?

Demo: Taking Derivatives with Vandermonde Matrices [cleared]
Finite Differences Numerically

Demo: Finite Differences [cleared]
Demo: Finite Differences vs Noise [cleared]
Demo: Floating point vs Finite Differences [cleared]
Taking Derivatives Numerically

Why *shouldn’t* you take derivatives numerically?

- \( \mathcal{D} \) is an unbounded operator

\[ \mathcal{D} : C^1 \rightarrow C^0 \]

\[ \| \mathcal{D} \| = \max_{\| f \|_\infty = 1} \| \mathcal{D} f \|_\infty \]

\[ \| A \| = \max_{\| x \| = 1} \| A x \| \]

\[ f_\lambda(x) = e^{i\lambda x} \quad \| f_\lambda \|_\infty = 1 \]

\[ \| \mathcal{D} f_\lambda \|_\infty = \| i\lambda e^{i\lambda x} \|_\infty \]

- Conditioning of \( \mathcal{D} \)

- Amplifies noise

- Loses order of accuracy vs. interp. \( h^n \) vs. \( h^{n-1} \)
\[ \Delta u_n = f_n \quad \text{and} \quad u_n \xrightarrow{n \to \infty} u \]
Find the order of accuracy of the finite difference formula
\[ f'(x) \approx \frac{f(x + h) - f(x - h)}{2h} \]
Outline

Introduction
- Notes
- Notes (unfilled, with empty boxes)
- About the Class
- Classification of PDEs
- Preliminaries: Differencing
- Interpolation Error Estimates (reference)

Finite Difference Methods for Time-Dependent Problems

Finite Volume Methods for Hyperbolic Conservation Laws

Finite Element Methods for Elliptic Problems

Discontinuous Galerkin Methods for Hyperbolic Problems
Truncation Error in Interpolation

If $f$ is $n$ times continuously differentiable on a closed interval $I$ and $p_{n-1}(x)$ is a polynomial of degree at most $n$ that interpolates $f$ at $n$ distinct points $\{x_i\}$ ($i = 1, \ldots, n$) in that interval, then for each $x$ in the interval there exists $\xi$ in that interval such that

$$f(x) - p_{n-1}(x) = \frac{f^{(n)}(\xi)}{n!}(x - x_1)(x - x_2)\cdots(x - x_n).$$
Truncation Error in Interpolation: cont’d.

\[ Y_x(t) = R(t) - \frac{R(x)}{W(x)} W(t) \quad \text{where} \quad W(t) = \prod_{i=1}^{n}(t - x_i) \]
What is the connection between the error result and Chebyshev interpolation?
Error Result: Simplified Form

Boil the error result down to a simpler form.

▶ Demo: Interpolation Error [cleared]
Outline

Introduction

Finite Difference Methods for Time-Dependent Problems
   1D Advection
   Stability and Convergence
   Von Neumann Stability
   Dispersion and Dissipation
   A Glimpse of Parabolic PDEs

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1D Advection Equation and Characteristics

\[ u_t + au_x = 0, \quad u(0, x) = g(x) \quad (x \in \mathbb{R}) \]

Solution?

Characteristic curve: Define \( x(t) \) so that
\[ u(x(t), t) = u(x(0), 0) \]

Suppose \( x \) is given by the IVP

\[
\begin{aligned}
\frac{dx}{dt} &= \frac{d}{dx}(u(x(t), t)) \\
\frac{dx}{dt} &= \frac{d}{dx}(u_0) \\
x(0) &= x_0
\end{aligned}
\]

\[
\frac{dn(x(t), t)}{dt} = u_x x'(t) + n_x = u_x \frac{d}{dx}(u(x(t), t)) + n_x = \frac{d}{dx}(u) + n_x = 0
\]
Solving Advection with Characteristics

\[ u_t + au_x = 0, \quad u(0, x) = g(x) \quad (x \in \mathbb{R}) \]

Find the characteristic curve for advection.

Generalize this to a solution formula.

Does the solution formula admit solutions that aren’t obviously allowed by the PDE?