HW1 due Friday
Consistency and Convergence

Assume $u, (\partial_x^{q_x})u, (\partial_t^{q_t})u \in L^2(\mathbb{R} \times [0, t^*])$.

**Definition (Consistency)**

A two-level scheme is **consistent** in the $L^2$-norm with order $q_t$ in time and $q_x$ in space if

$$\max_{l \leq h_x \leq h^*} \| e^l_t \| = O(h_x^{q_x} + h_t^{q_t}) \quad \text{as} \quad h_x \to 0 \quad h_t \to 0$$

**Definition (Convergence)**

A two-level scheme is **convergent** in the $L^2$-norm with order $q_t$ in time and $q_x$ in space if

$$\max_{l \leq h_x \leq h^*} \| e^l \| = O(h_x^{q_x} + h_t^{q_t}) \quad \text{as} \quad h_x \to 0 \quad h_t \to 0$$
Stability

\[ P_h v_{\ell+1} = Q_h v_\ell \]

Write down a matrix product to bring \( v_0 \) to \( v_\ell \):

\[ \tilde{v}_\ell = (P_h^{-1} Q_h)^\ell \tilde{v}_0 \]

Definition (Stability)

A two-level scheme is \textbf{stable} in the \( L^2 \)-norm if there exists a constant \( c > 0 \) independent of \( h_t \) and \( h_x \) so that

\[ \left\| (P_h^{-1} Q_h)^\ell P_h^{-1} \right\| \leq c \]

for all \( \ell \) and \( h_t \) such that \( \ell h_t \leq t^\ast \).
Lax Convergence Theorem

Theorem (Lax Convergence)

If a two-level FD scheme is

- consistent in the $L^2$-norm with order $q_t$ in time and $q_x$ in space, and
- stable in the $L^2$-norm, then

it is convergent in the $L^2$-norm with order $q_t$ in time and $q_x$ in space.
Lax Convergence: Proof (1/2)

\[ p_h \tilde{e}_{n+1} = Q_h \tilde{e}_n + \frac{2}{h} \ell \eta \]

\[ \tilde{e}_{n+1} = p_h^{-1} Q_h \tilde{e}_n + p_h^{-1} \ell \eta \]

Recall: \(	ilde{e}_0 = 0\) (assumption)

\[ \tilde{e}_1 = \ell h \left( p_h^{-1} \tilde{e}_0 \right) \]

\[ \tilde{e}_2 = h \left( h^{-1} Q_h \right) p_h^{-1} \tilde{e}_0 + p_h^{\ell} \tilde{e}_1 \]

By induction:

\[ \tilde{e}_n = h_k \left( \sum_{m=1}^{\ell-m} p_h^{-1} Q_h \right) \tilde{e}_{m-1} \]
Lax Convergence: Proof (2/2)

\[ e_\ell = h_t \sum_{m=1}^{\ell} (P_h^{-1}Q_h)^{\ell-m} P_h^{-1} \tau_{m-1}. \]

\[ \| A \times \| \leq A \| A \| \| x \| \]

\[ 0 \leq h_t \leq \epsilon^\ell \]

\[ \| e_\ell \| \leq h_t \sum_{m=1}^{\ell} \| (P_h^{-1}Q_h)^{\ell-m} P_h^{-1} \| \| \tau_{m-1} \| \]

\[ \leq h_t \sum_{m=1}^{\ell} \| (P_h^{-1}Q_h)^{\ell-m} P_h^{-1} \| \| \tau_{m-1} \| \leq C \text{ (stab)} \]

\[ \leq h_t \sum_{m=1}^{\ell} \| \tau_{m-1} \| \leq C \max_{\ell} \| \tau_{\ell} \| = C \epsilon < O(h_t^{n_x} + h_t^{n_z}) \]

\[ \forall h_t \leq \epsilon^{\ell} \land \| x \| = O(h_t^{n_x} + h_t^{n_z}) \]

\[ = O(h_t^{n_x} + h_t^{n_z}) \text{ consistency} \]
Conditions for Stability

\[ \left\| (P_h^{-1} Q_h)^\ell P_h^{-1} \right\| \leq c \]

Give a simpler, sufficient condition:

\[ \| P_h^{-1} Q_h \| \leq 1 \quad \| P_h^\dagger \| \ll c \]

\( \Leftrightarrow \) Lax-Richtmyer stability

How can we show bounds on these matrix norms?

- bounds have to hold for all \( h_x, h_t \)
- proving this: generally cumbersome
- to prove: bound singular values
Stability of ETBS (1/3)

Theorem (Gershgorin)

For a matrix \( A \in \mathbb{C}^{N \times N} = (a_{i,j}) \),

\[
\sigma(A) \subset \bigcup_{j=1}^{N} \bar{B} \left( a_{j,j}, \sum_{k \neq j} |a_{j,k}| \right).
\]

The spectrum (all eigenvalues)

ETBS:

\[
\frac{u_{k,\ell+1} - u_{k,\ell}}{h_t} + a \frac{u_{k,\ell} - u_{k-1,\ell}}{h_x} = 0
\]

Analyze stability of ETBS:

Let \( \lambda = \frac{ah_t}{h_x} \), \( u_{k,\ell+1} = \lambda u_{k-1,\ell} + (1-\lambda) u_{k,\ell} \).

Let \( P_n = I \) and \( Q_n = \text{tridiag} \left( \lambda, 1-\lambda, 0 \right) \). \( \|P_n\| \leq 1 \).
Consider singular values of $P_h^{-1}Q_h = Q_h$; eigenvalues of $Q_h^TQ_h$.

$Q_h^TQ_h = \text{tridiag}(λ(1−λ), (1−λ)^2 + λ^2, λ(1−λ))$. Assume $0 ≤ λ ≤ 1$.

$⇒ λ(1−λ) ≥ 0$. Let $λ$ be an eigenvalue of $Q_h^TQ_h$.

$2λ^2 - 2λ ≤ λ − (1−λ)^2 − λ^2 ≤ 2λ^2 - 2λ^2 ≤ 0$

$λ - 1 + 2λ - 2λ^2 ≤ 0$

$1 - λ + 2λ^2 ≤ λ ≤ 1$

$0 ≤ (2λ - 1)^2 = λ ≤ 1$

"Analogously," if $λ \notin [0,1]$, $|λ|$ is bounded above by 1.
Stability of ETBS (3/3)

Summarize ETBS stability:

ETBS is stable if and only if $0 \leq \lambda \leq 1$

"conditional stability"

$0 \leq \frac{ah_{e}}{h_{x}} \leq 1 \Leftrightarrow h_{e} \leq \frac{h_{x}}{a}

Courant-Friedrichs-Lewy condition, "CFL" condition

Comments?