Consider only $f(u) = au$ for now. Riemann solver inspiration from FD?

\[
\begin{align*}
\frac{\partial u}{\partial t} + a u \frac{\partial u}{\partial x} &= 0 \\
\text{const. coeff.} \\
\text{var. coeff.}
\end{align*}
\]

\[
0 = \frac{u_{j+1} - u_{j-1}}{h_x} + \frac{f^*(u_{j,1}, u_{j,1}) - f^*(u_{j-1,1}, u_{j-1,1})}{h_x}
\]

\[
f^*(u^-, u^+) = \begin{cases}
    au^- & \text{a} > 0 \\
    au^+ & \text{a} < 0
\end{cases}
\]

\[
\frac{au^- + au^+}{2} - \frac{|a|}{2} (u^+ - u^-)
\]

\[
\text{var. coeff.: which a?}
\]
Side Note: First Order Upwind, Rewritten

\[
\begin{align*}
\frac{u_{j,\ell+1} - u_{j,\ell}}{h_t} &+ \frac{f^*(u_{j,\ell}, u_{j+1,\ell}) - f^*(u_{j-1,\ell}, u_{j,\ell})}{h_x} \\
\text{with} & \\
 f^*(u^-, u^+) = \frac{a u^- + a u^+}{2} - \frac{|a|}{2} (u^+ - u^-).
\end{align*}
\]

\[
\frac{u_{j,\ell+1} - u_{j,\ell}}{h_t} + a \frac{u_{j+1,\ell} - u_{j-1,\ell}}{2h_x} = \frac{ah_x}{2} \left( \frac{u_{j+1,\ell} - 2u_{j,\ell} + u_{j-1,\ell}}{h_x^2} \right) \in \mathcal{C}_5
\]

\[
= u_{x,xx} + o(h_x^3)
\]
Lax-Friedrichs generalize linear upwind flux for a nonlinear conservation law:

\[ f^*(u^-, u^+) = \frac{au^- + au^+}{2} - \frac{|a|}{2}(u^+ - u^-). \]

**Demo:** Finite Volume Burgers [cleared] (Part I)
Attempt 2: \[ \Delta = \max \left( \left| \frac{\partial u}{\partial x} \right|, \left| \frac{\partial u}{\partial y} \right| \right) \]  

Local Lax-Friedrichs flux/Rusanov

'global L-F': take max over whole solution

'global': more dissipative
Outline

Introduction

Finite Difference Methods for Time-Dependent Problems

Finite Volume Methods for Hyperbolic Conservation Laws
  Theory of 1D Scalar Conservation Laws
  Numerical Methods for Conservation Laws
  Higher-Order Finite Volume
  Outlook: Systems and Multiple Dimensions

Finite Element Methods for Elliptic Problems

Discontinuous Galerkin Methods for Hyperbolic Problems
Improving Accuracy

Consider our existing discrete FV formulation:

\[ \bar{u}_{j,\ell+1} = \bar{u}_{j,\ell} - \frac{h_t}{h_x} \left( f(u_{j+1/2,\ell}) - f(u_{j-1/2,\ell}) \right). \]

What obstacles exist to increasing the order of accuracy?

- Time discre.
- Spatial discre.
- Non-smoothness

What order of accuracy can we expect?

Near a discontinuity: won’t be able to do better than first order. Goal: more accuracy away from the shock.
Improving the Order of Accuracy

Improve temporal accuracy.

\[ \frac{d\hat{u}_j(t)}{dt} + \frac{\partial}{\partial x} \left( f(u_{j-\frac{1}{2}}, u_{j+\frac{1}{2}}) - g(u_{j-\frac{1}{2}}, u_{j+\frac{1}{2}}) \right) = 0 \]

\[ \Rightarrow \text{Idea: use higher-order method on that ODE} \]

What’s the obstacle to higher spatial accuracy?

Letting \( u_{j-\frac{1}{2}} = \bar{u} \), \( u_{j+\frac{1}{2}} = \bar{u} \),

How can we improve the accuracy of that approximation?

Involving more cells in the reconstruction!
Increasing Spatial Accuracy

Temporary Assumptions:

- $f'(u) \geq 0$
- $f_{j+1/2}^* = f(\bar{u}_j)$ (e.g. Godunov in this situation)

Reconstruct $u_{j+1/2}$ using $\{\bar{u}_j - 1, \bar{u}_j, \bar{u}_{j+1}\}$. Accuracy? Names?

Compute fluxes, use increments over cell average:
\[ u = ax + b \]

\[
\frac{1}{h_x} \int_{-\frac{h}{2}}^{\frac{h}{2}} ax + b = \tilde{u}_{j-1}
\]

\[
\frac{1}{h_y} \int_{-\frac{h}{2}}^{\frac{h}{2}} ax + b = \tilde{u}_j
\]

\[ u \left( \frac{h}{t} \right) = \]
Lax-Wendroff

For $u_t + au_x$, from finite difference:

$$f^*(u^-, u^+) = \frac{au^- + au^+}{2} - \frac{a^2}{2} \cdot \frac{\Delta t}{\Delta x} (u^+ - u^-).$$

Taylor in time: $u_{\ell+1} = u_\ell + \partial_t u_\ell \cdot h_t + \partial^2_t u_\ell \cdot h_t/2 + O(h_t^3)$.

$$\frac{u_{j,\ell+1} - u_{j,\ell}}{h_t} + \frac{f(u_{j+1,\ell}) - f(u_{j-1,\ell})}{2h_x}$$

$$= \frac{h_t}{2h_x} \left[ f'(u_{j+1/2,\ell}) \frac{f(u_{j+1,\ell}) - f(u_{j,\ell})}{h_x} - f'(u_{j-1/2,\ell}) \frac{f(u_{j,\ell}) - f(u_{j-1,\ell})}{h_x} \right]$$

As a Riemann solver:

$$f^*(u^-, u^+) = \frac{f(u^-) + f(u^+)}{2} - \frac{h_t}{h} \left[ f'(u^o)(f(u^+) - f(u^-)) \right].$$