Definition 5.7: Two-Level Linear Finite-Difference Scheme

A finite-difference scheme that can be written as,

$$P_h \boldsymbol{u}_{\ell+1} = Q_h \boldsymbol{u}_\ell + h_t \boldsymbol{b}_\ell, \tag{5.5}$$

is called a two-level linear finite-difference scheme. Each iteration depends only on two instances of time. Examples are given in Example 5.8.

ETBS: $$\frac{u_{k,\ell+1} - u_{k,\ell}}{h_t} + a\,\frac{u_{k,\ell} - u_{k-1,\ell}}{h_x} = 0$$

$$u_{k,\ell+1} - u_{k,\ell} + \frac{a h_t}{h_x}\left(u_{k,\ell} - u_{k-1,\ell}\right) = 0$$

$$u_{k,\ell+1} = \left(1 - \frac{a h_t}{h_x}\right) u_{k,\ell} + \frac{a h_t}{h_x} u_{k-1,\ell}$$

$$\Longrightarrow \quad P_h = I$$

$$Q_h = \operatorname{tridiag}\left(\frac{a h_t}{h_x},\ 1 - \frac{a h_t}{h_x},\ 0\right)$$

$u_{k+1,\ell}$

Definition 5.10: Truncation Error

The local truncation error, $\tau_{k,\ell}$, is the error that remains when a finite-difference method is applied to the exact solution, $u(x_k, t_\ell)$.

Example 5.12: ETFS Truncation Error

$$\begin{aligned}
\tau_{k,\ell} =& \frac{u(x_k, t_{\ell+1}) - u(x_k, t_\ell)}{h_t} + a\frac{u(x_{k+1}, t_\ell) - u(x_k, t_\ell)}{h_x} \\
=& \frac{1}{h_t}\left(u(x_k, t_\ell) + u_t(x_k, t_\ell)h_t + u_{tt}(x_k, \varsigma)\frac{h_t^2}{2} - u(x_k, t_\ell)\right) \\
&+ \frac{a}{h_x}\left(u(x_k, t_\ell) + u_x(x_k, t_\ell)h_x + u_{xx}(\xi^+, t_\ell)\frac{h_x^2}{2} - u(x_k, t_\ell)\right) \\
=& u_{tt}(x_k, \varsigma)\frac{h_t}{2} + a u_{xx}(\xi^+, t_\ell)\frac{h_x}{2} \\
=& \mathcal{O}(h_t, h_x)
\end{aligned}$$

Example 5.14: ETCS Truncation Error (Matrix Form)

For ETCS, $P_h = I$ and

$$\begin{aligned}(P_h \boldsymbol{U}_{\ell+1})_k &= u(x_k, t_{\ell+1}) = u(kh_x, (\ell+1)h_t),\\ (Q_h \boldsymbol{U}_\ell)_k &= u(kh_x, \ell h_t) - \frac{ah_t}{2h_x}\Big(u((k+1)h_x, \ell h_t) - u((k-1)h_x, \ell h_t)\Big).\end{aligned}$$

Thus,

$$\begin{aligned}(P_h \boldsymbol{U}_{\ell+1} - Q_h \boldsymbol{U}_\ell)_k =& u_t(kh_x, \ell h_t)h_t + u_{tt}(kh_x, \varsigma)\frac{h_t^2}{2} + ah_t u_x(kh_x, \ell h_t)\\ &+ \frac{ah_t}{12}\Big(u_{xxx}(\xi^+, \ell h_t) + u_{xxx}(\xi^-, \ell h_t)\Big)h_x^2\\ =& \Big(u_{tt}(kh_x, \varsigma)\frac{h_t}{2} + \frac{a}{12}(u_{xxx}(\xi^+, \ell h_t) + u_{xxx}(\xi^-, \ell h_t))h_x^2\Big)h_t\\ =& \tau_{k,\ell} h_t.\end{aligned}$$

In general, $P_n U_{\ell+1} = Q_n U_\ell + b_\ell h_t + \tau_\ell h_t$

$$\tau_\ell = \begin{bmatrix} \tau_{-1\ell} \\ \tau_{0\ell} \\ \tau_{1\ell} \end{bmatrix}$$

$\underline{b=0}$

$$P_n U_{\ell+1} = Q_n U_\ell + \tau_\ell h_t$$

$$P_n u_{\ell+1} = Q_n u_\ell$$

$$\Rightarrow P_n e_{\ell+1} = Q e_\ell + \tau_\ell h_t$$

$$\Rightarrow e_{\ell+1} = P_n^{-1} Q_n u_\ell + P_n^{-1} \tau_n h_{t,}$$

- the max-norm (ℓ_∞): $\|\boldsymbol{e}\|_\infty = \max\limits_{k,\ell} |e_{k,\ell}|$; or
- the scaled Euclidean norm (ℓ_2): $\|\boldsymbol{e}\|_2 = \left(\sum\limits_k \sum\limits_\ell h_x h_t (e_{k,\ell})^2 \right)^{\frac{1}{2}}$

Demo

$nx = 64$

$\downarrow$

$nx = 128$

Definition 5.15: Consistency, Stability, and Convergence

Let $\frac{\partial^\mu u}{\partial x^\mu}$ *denote the* μ*-th partial derivative in* x*, and* $\frac{\partial^\nu u}{\partial t^\nu}$ *denote the* ν*-th partial derivative in* t*. Assume that* $\frac{\partial^\mu u(x,\hat{t})}{\partial x^\mu}, \frac{\partial^\nu u(x,\hat{t})}{\partial t^\nu} \in L^2(\mathbb{R})$*, for all* $\hat{t} < t^*$*.*

A two-level linear finite-difference scheme, $P_h \boldsymbol{u}_{\ell+1} = Q_h \boldsymbol{u}_\ell + \boldsymbol{b}_\ell h_t$*, is*

- *consistent in the* L^2*-norm with order* ν *in time and* μ *in space if*

$$\max_{\ell, \ell h_t \le t^*} \|\boldsymbol{\tau}_\ell\| = \mathcal{O}(h_x^\mu, h_t^\nu);$$

- *convergent in the* L^2*-norm with order* ν *in time and* μ *in space if*

$$\max_{\ell, \ell h_t \le t^*} \|\boldsymbol{e}_\ell\| = \mathcal{O}(h_x^\mu, h_t^\nu);$$

- *stable in the* L^2*-norm if* $\exists c > 0$*, independent of* h_t *and* h_x*, such that* $\|(P_h^{-1} Q_h)^\ell P_h^{-1}\| \le c$ *for all* ℓ *and* h_t *such that* $\ell h_t \le t^*$*.*

- consistent if the scheme is "right" for the PDE.
- convergence is a statement on the error.

Linear algebra recall.

Solve $Ax=b$

→ a stable algorithm gives a small residual: $r = b - Ax$

But

$$\frac{\|\Delta x\|}{\|x\|} \leq \operatorname{cond}(A) \frac{\|r\|}{\|A\| \|x\|}$$

How do we use this?

Two tools:

Theorem 5.19: Lax Convergence Theorem

If a two-level linear finite-difference scheme, $P_h \boldsymbol{u}_{\ell+1} = Q_h \boldsymbol{u}_\ell + \mathbf{b}_\ell h_t$, is consistent *in the L^2-norm with order ν in time and μ in space* ***and*** stable *in the L^2-norm, then it is* convergent *in the L^2-norm with order ν in time and μ in space.*

Stronger :

Theorem 5.20: Lax Equivalence Theorem

Let the two-level linear finite-difference scheme, $P_h \boldsymbol{u}_{\ell+1} = Q_h \boldsymbol{u}_\ell + h_t \boldsymbol{b}_\ell$, be consistent in the L^2-norm with order ν in time and μ in space, then it is stable in the L^2-norm if and only if *it is convergent in the L^2-norm with order ν in time and μ in space.*

Main point

consistency + stability $\Rightarrow$ convergence

easy

hard

Norms

Consider matrix T.

What is $\|T\|$?

$$\|T\| = \max_i |\sigma_i(T)|$$

singular values

also $\|T\| = \sup \dfrac{\|Tx\|}{\|x\|}$

$$\|Tx\| \leq \|T\| \, \|x\|$$

How to establish stability

Need $\|(P_n^{-1} Q_n)^\ell P_n^{-1}\| \leq C$ $\forall \ell$

$\forall \ell h < t^*$

$$\Downarrow$$

$$\|(P_h^{-1} Q)^\ell\| \, \|P_h^{-1}\|$$

So we need $\|P^{-1} Q\| < 1$

and $\|P^{-1}\| < C$

"Lax-Richtmyer" stability

ETBS

$$\frac{u_{k\,l+1} - u_{kl}}{h_t} + a\,\frac{u_{kl} - u_{k-1\,l}}{h_x} = 0$$

$$\gamma = \frac{a h_t}{h_x}$$

$$\rightarrow \quad u_{k\,l+1} = \gamma u_{k-1\,l} + (1-\gamma)\,u_{kl}$$

$$\rightarrow \quad P_h = I$$

$$Q_h = \mathrm{tridiag}(\gamma,\ 1-\gamma,\ 0)$$

Consider $u(0,t) = f(t)$ ~~as~~ a boundary condition

First P_n: $P = I \Rightarrow \|P^{-1}\| = 1$

Next $P_n^{-1} Q_n = Q_n$:

consider $\|Q_n\|_1 = \max\limits_{1 \le j \le n} \sum\limits_{i=1}^{n} |(Q_n)_{i,j}|$

or

$\|Q_n\|_\infty = \max\limits_{i} \sum\limits_{j=1}^{n} |(Q_n)_{i,j}|$

$\|Q\|_1 = \|Q\|_\infty = |\gamma| + |1-\gamma|$

if $0 \le \gamma \le 1$

then $\|Q\|_1 = 1$

if $\gamma \notin [0,1]$

then $\|Q\|_1 > 1$

(similar for ℓ_2 using Geršgorin's Thm)

$\Rightarrow$ conditional stability

if $0 \le \gamma \le 1$, then stable.

a>0

$\Rightarrow \quad \frac{a h_t}{h_x} \le 1 \Rightarrow h_t \le \frac{h_x}{a}$

CFL condition

Ok this is complicated.

Definition 5.28: Symbol of a Two-Level Finite-Difference Scheme

Given a two-level linear finite-difference scheme, $P_h \boldsymbol{u}_{\ell+1} = Q_h \boldsymbol{u}_\ell + \boldsymbol{b}_\ell h_t$, such that p_k and q_k are the coefficients of the scheme corresponding to P_h and Q_h, respectively, then,

$$\hat{\boldsymbol{p}}(\varphi) = \sum_k p_k e^{-i\varphi k} \text{ and } \hat{\boldsymbol{q}}(\varphi) = \sum_k q_k e^{-i\varphi k}.$$

The symbol of the two-level finite-difference method is $s(\varphi) = \dfrac{\hat{\boldsymbol{q}}(\varphi)}{\hat{\boldsymbol{p}}(\varphi)}$.

Definition 5.29: von Neumann Stability

Consider a two-level linear finite-difference scheme, $P_h \boldsymbol{u}_{\ell+1} = Q_h \boldsymbol{u}_\ell + \boldsymbol{b}_\ell h_t$, with symbol, $s(\varphi)$. If

$$\max_\varphi |s(\varphi)| \le 1 \qquad \text{and} \qquad \max_\varphi \left| \frac{1}{\hat{\boldsymbol{p}}(\varphi)} \right| \le c,$$

for some constant $c > 0$, then we say the scheme is von Neumann stable.

What is p_k, q_k? Why? next time.