Today 1/30

Stability, dissipation, dispersion.

Recap

① Two-level schemes

ignore

$$P u_{\ell+1} = Q u_\ell + \cancel{h_t \underline{b}_\ell}$$

↑ time $\ell+1$ ↑ time ℓ

② $\underline{u}$ is the numerical approx

$\underline{\bar{u}}$ is the exact.

$$\underline{e} = \bar{u} - u$$

③ $\underline{\tau}$ is the truncation error

is "the remainder after applying the FD scheme to the exact solution"

$$\Rightarrow P\,\overline{u}_{\ell+1} = Q\,\overline{u}_{\ell} + \tau h_t$$

why?

Example 5.14

④ How do errors propagate?

$$P \mathcal{U}_{\ell+1} = Q \mathcal{U}_\ell + \tau h_t$$

$$- \left(P u_{\ell+1} = Q u_\ell \right)$$

$$\Rightarrow P e_{\ell+1} = Q e_\ell + \tau h_t$$

$$e_0 = 0 \quad (\text{since } u_0 = \mathcal{U}_0)$$

$$P e_1 = Q e_0 + \tau_0 h_t$$

$$= \tau_0 h_t$$

$$e_1 = P^{-1} \tau_0 h_t$$

$$e_1 = P^{-1} \tau_0 h_t$$

$$\overline{P} e_2 = Q e_1 + \tau_1 h_t$$

$$e_2 = P^{-1} Q e_1 + P^{-1} \tau_1 h_t$$

$$= P^{-1} Q P^{-1} \tau_0 h_t + P^{-1} \tau_1 h_t$$

$$\overline{P} e_3 = \cdots$$

$$P e_4 = \cdots$$

$$\vdots$$

$$e_\ell = \sum_{m=1}^{\ell} (P^{-1} Q)^{\ell - m} P^{-1} \tau_\ell h_t + P^{-1} \tau_0 h_t$$

(5) $$e_\ell = h_t \sum_{m=1}^{\ell} (P^{-1}Q)^{\ell-m} P^{-1} \tau_{m-1}$$

$$\Rightarrow \|e_\ell\| \le h_t \sum_{m=1}^{\ell} \underbrace{\|(P^{-1}Q)^{\ell-m} P^{-1}\|}_{\substack{\text{stability} \\ \le C}} \underbrace{\|\tau_{m-1}\|}_{\substack{\text{Consistent} \\ \to 0 \\ \text{as} \\ h_t, h_x \to 0}}$$

(6) A scheme is consistent if $\tau_\ell \to 0$ as $h_t, h_x \to 0$

(7) A scheme is convergent if $e_\ell \to 0$ as $h_t, h_x \to 0$.

(7) A scheme is stable if

$$\|(P^{-1}Q)^{\ell} P^{-1}\| \le C \quad \text{for some } c,$$

⑧ How to show stability?

① bound the matrix norms ✓

② von Neumann stability

V.N. stability

Short story (Section 5.3.2 in text)

- P and Q are Toeplitz
- Can write $(Px)_j = \sum_k p_{j-k} x_k$

$(Qx)_j = \sum_k q_{j-k} x_k$

- ... $\|P^{-1}Q\| = \max \left| \dfrac{\hat{q}(\phi)}{\hat{p}(\phi)} \right|$

Fourier Trans. of $\underset{\sim}{p}, \underset{\sim}{q}$

if $\left| \dfrac{\hat{q}(\phi)}{\hat{p}(\phi)} \right| < 1$

then stable

In practice:

let $u_{k,\ell} = \lambda^{\ell} e^{ik\theta}$

$$\gamma = \frac{a h_t}{h_x}$$

Example

$$u_{k,\ell+1} = \gamma u_{k-1,\ell} + (1-\gamma) u_{k,\ell}$$

$$\lambda^{\cancel{\ell+1}} \cancel{e^{ik\theta}} = \gamma \cancel{\lambda^{\ell}} \cancel{e^{ik\theta}} e^{-i\theta} + (1-\gamma) \cancel{\lambda^{\ell}} \cancel{e^{ik\theta}}$$

$$\lambda = \gamma e^{-i\theta} + (1-\gamma)$$

Find conditions so that

$$\left|\frac{\hat{q}}{\hat{p}}\right| = |\lambda| \leq 1$$

$$\lambda = \gamma e^{-i\theta} + (1-\gamma)$$

$$|\lambda|^2 \le 1 ?$$

$$\lambda = 1 - \gamma(1 - e^{-i\theta})$$

$$\Rightarrow |\lambda|^2 = 1 - \gamma(1 - e^{-i\theta}) - \gamma(1 - e^{i\theta}) + \gamma^2 \cancel{|}1 - e^{i\theta}\cancel{|}^2$$

$$= 1 - 2\gamma + 2\gamma\cos(\theta) + \gamma^2 \underbrace{\left(\underbrace{1 - e^{-i\theta} - e^{i\theta}}_{1 - 2\cos\theta} + \underbrace{e^{-i\theta + i\theta}}_{1}\right)}_{2 - 2\cos\theta}$$

$$= 1 - 2\gamma + 2\gamma\cos\theta + \gamma^2(2 - 2\cos\theta)$$

$$|\lambda|^2 = 1 - 2\gamma + 2\gamma^2 + 2(\gamma - \gamma^2)\cos\theta$$

take $\frac{\partial}{\partial\theta}$

$$\rightarrow \frac{\partial|\lambda|^2}{\partial\theta} = -2(\gamma - \gamma^2)\sin\theta \equiv 0$$

if $\theta = m\pi$

$m \in \mathbb{Z}$

$$= 1 - 2\gamma + 2\gamma^2 + 2(\gamma - \gamma^2)(\pm 1)$$

if m is even:

$$|\lambda|^2 = 1 - 2\gamma + 2\gamma^2 + 2\gamma - 2\gamma^2$$
$$= 1$$

if m is odd

$$|\lambda|^2 = 1 - 4\gamma + 4\gamma^2 = |1 - 2\gamma|^2$$

$$|\lambda|^2 = |1-2\gamma|^2 \leq 1$$

iff

$$0 \leq \gamma \leq 1$$

or

$$0 \leq \frac{a h_t}{h_x} \leq 1$$

or

$$h_t \leq \frac{h_x}{a}$$

$$u_t + a u_x = 0$$

Can we do better?

Taylor $\quad u(x,t+h_t) = u(x,t) + h_t\, u_t(x,t)$

$$+ \frac{h_t^2}{2} u_{tt}(x,t) + O(h_t^3)$$

$u_t + a u_x = 0$

$\rightarrow u_t = -a u_x$

also

$u_{tt} = -a u_{tx}$

$\rightarrow u_{tt} = a^2 u_{xx}$

$$u(x,t+h_t) = u + h_t u_x + \frac{h_t^2}{2} u_{xx}$$

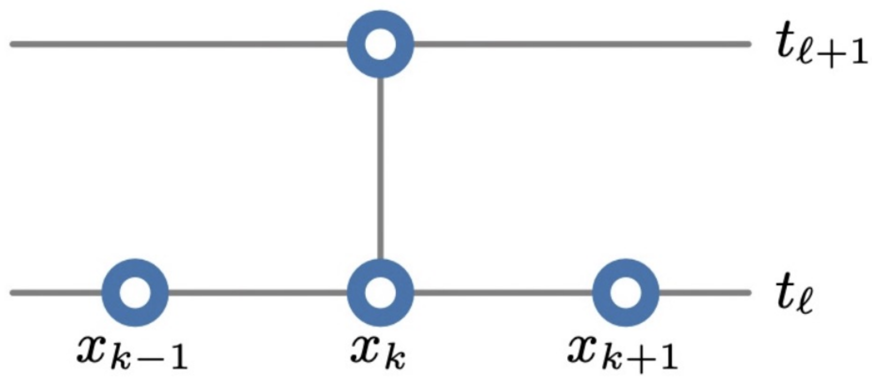
centered $\qquad$ centered

explicit time

centered

2nd order u_{xx}

$$\frac{u_{k,\ell+1} - u_{k,\ell}}{h_t} + a\frac{u_{k+1,\ell} - u_{k-1,\ell}}{2h_x} - \frac{a^2 h_t}{2}\frac{u_{k+1,\ell} - 2u_{k,\ell} + u_{k-1,\ell}}{h_x^2} = 0$$



f. Lax-Wendroff (LW)