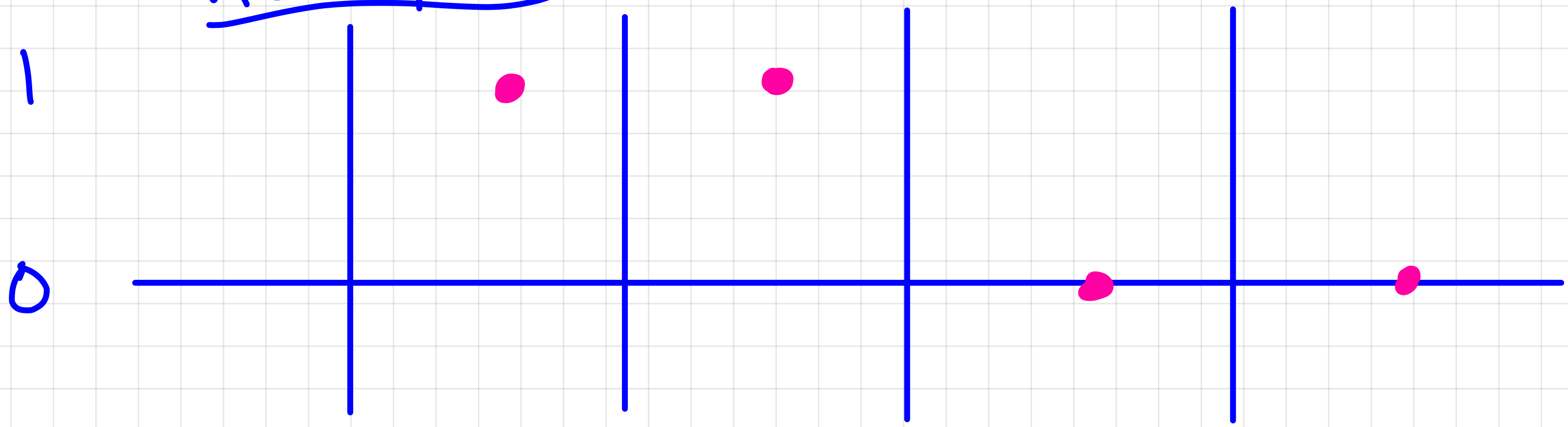
Today 2/15

Linear reconstruction (higher resolution)

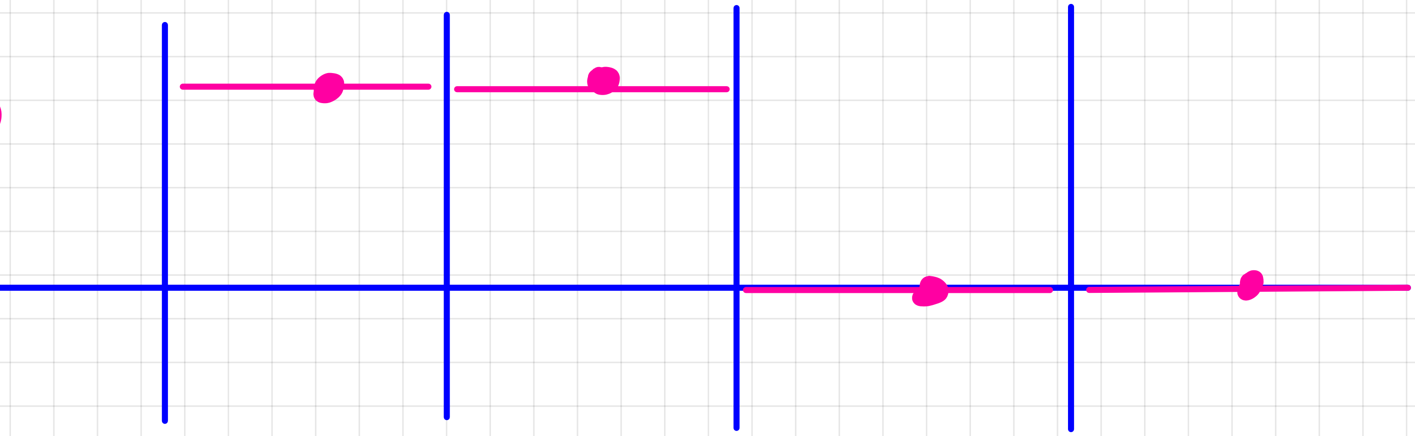
Recap



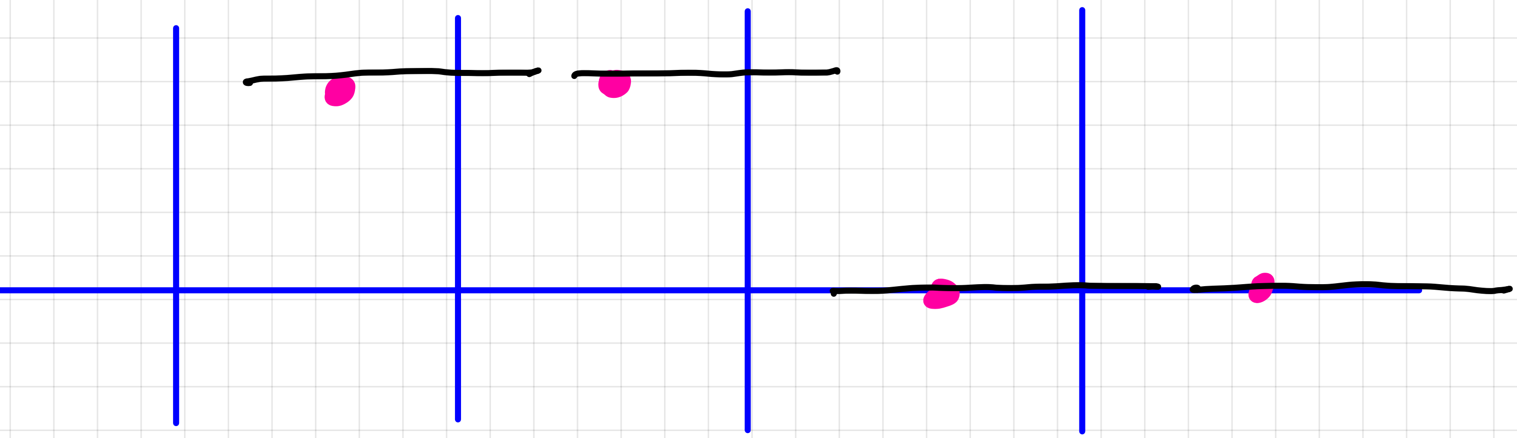
1. Reconstruct piecewise constant $u_e(x)$
2. Evolve all Riemann problems h_t
3. Compute averages

GOTO (1)

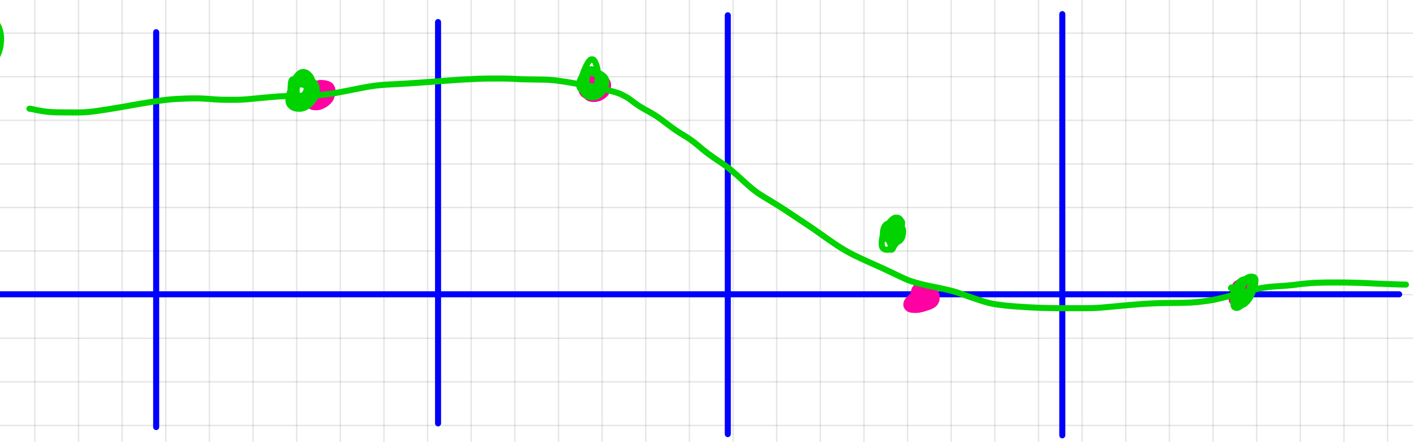
1



2



3



What next?

- [x] linear, nonlinear scalar conservation laws.

$$u_t + (f(u))_x = 0$$

"first order"

- For higher order (higher resolution)

nonlinear: Godunov - could approximate $f(\cdot)$ with linear (Roe's)

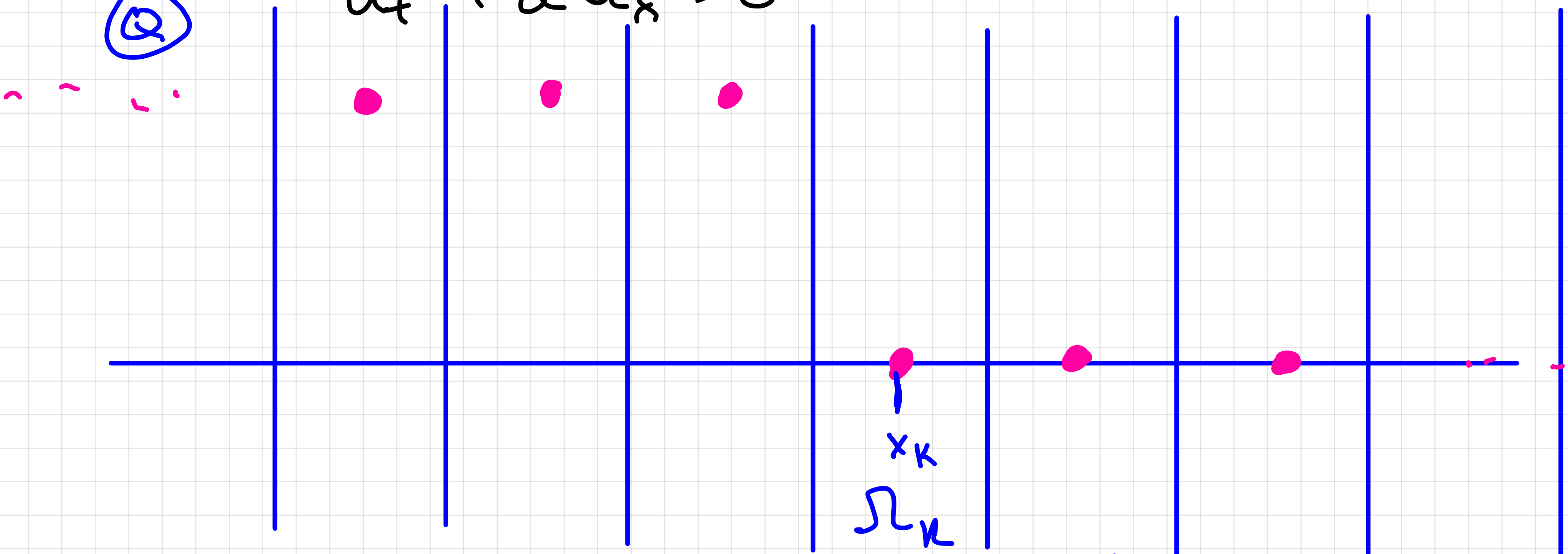
linear: linear reconstruction + REA

R ← reconstruct
E ← evolve
A ← average

- systems . . . ~ upcoming
- 2D/3D? . . . ~ upcoming

Q

$u_t + a u_x = 0$



reconstruct the piecewise function as pw linear:

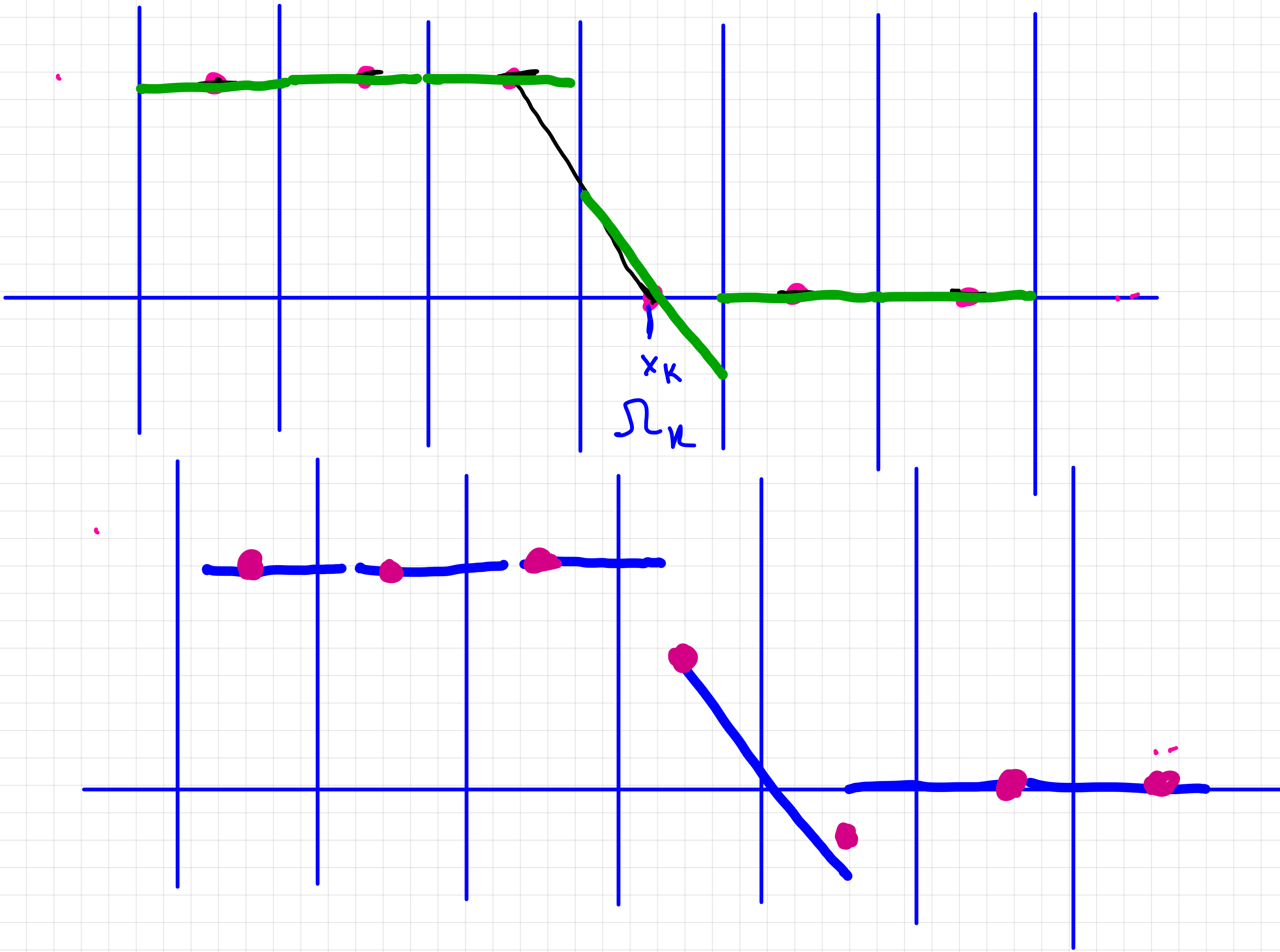
$$q_{k,\ell} = u_{k,\ell} + \frac{u_{k\ell} - u_{k-1,\ell}}{h_x}(x - x_k)$$

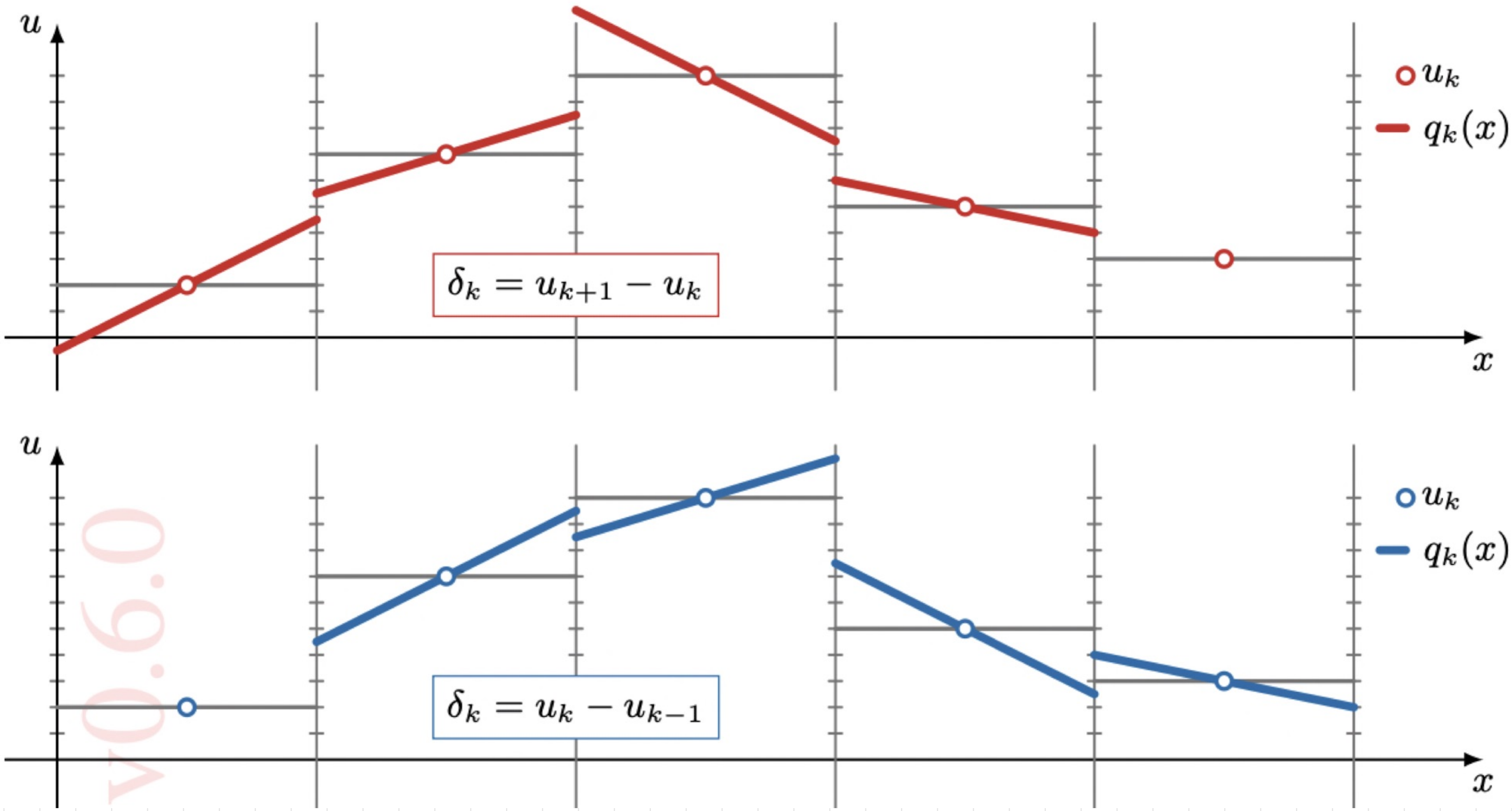
avg. values

$$\int_{\Omega_k} q_\ell(x)\,dx = u_{k,\ell}$$

$$u_k + \frac{u_k - u_{k-1}}{h_x}(x - x_k)$$

slope, δ_k


x_K
Ω_K



u
u_k
$q_k(x)$
$\delta_k = u_{k+1} - u_k$
x
u
u_k
$q_k(x)$
$\delta_k = u_k - u_{k-1}$
x

In practice

$$(*) \quad u_{k\ell+1} = u_{k\ell} + \cdots\cdots$$

or

$$\frac{u_{k\ell+1} - u_k}{h_t} + \frac{f^*_{k+1/2} - f^*_{k-1/2}}{h_x} = 0$$

① u_k = average → reconstruct

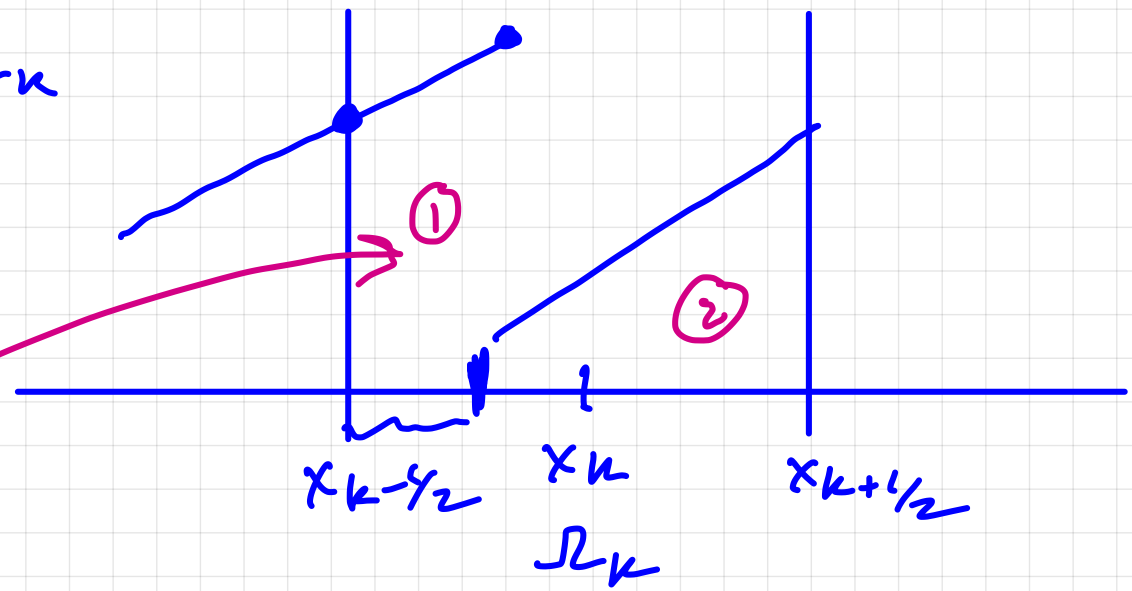
$$q_k(x) = u_k + \delta_k (x - x_k)$$

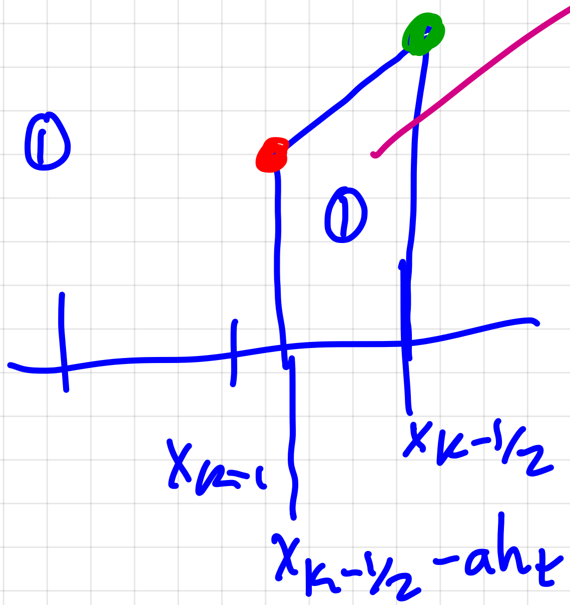
free param.

$$\text{let } u(x, t_\ell) = u_{k\ell} + \delta_{k\ell}(x - x_k)$$

② $$u(x, t_{\ell+1}) = u(x - at, t_\ell)$$

③ $u_{k,\ell+1} = \frac{1}{\Delta x}\int_{\Omega_k} u(x-at, t_\ell)\,dx$





@ $x = x_{k-1/2} - ah_t$:

$$u_{k-1,\ell} + \delta_{k-1,\ell}\left(x_{k-1/2} - ah_t - x_{k-1}\right)$$

@ $x = x_{k-1/2}$:

$$u_{k-1,\ell} + \delta_{k-1,\ell}\left(x_{k-1/2} - x_{k-1}\right)$$

$$\to \frac{1}{h_x}\int_{x_{k-1/2}-ah_t}^{x_{k-1/2}} ① = \frac{ah_t}{2h_x}\left[u_{k-1,\ell} + \delta_{k-1,\ell}\left(\frac{h_x}{2} - ah_t\right) + u_{k-1,\ell} + \delta_{k-1,\ell}\left(\frac{h_x}{2}\right)\right]$$

① : $\dfrac{ah_t}{h_x}\left[u_{k-1,\ell} + \dfrac{h_x - ah_t}{2}\,\delta_{k-1,\ell}\right]$

$\vdots$

② $\dfrac{h_x - ah_t}{2h_x}\left[u_{k,\ell} - \dfrac{ah_t}{2}\,\delta_{k\ell}\right]$

$\xrightarrow{\text{Combine}}$

$$u_{k,\ell+1} = u_{k\ell} - \frac{ah_t}{h_x}\left(u_{k\ell} - u_{k-1\ell}\right) - \frac{ah_t}{h_x}\left(h_x - ah_t\right)\left(\delta_{k\ell} - \delta_{k-1\ell}\right)$$

$\sigma = 0$ Godunov

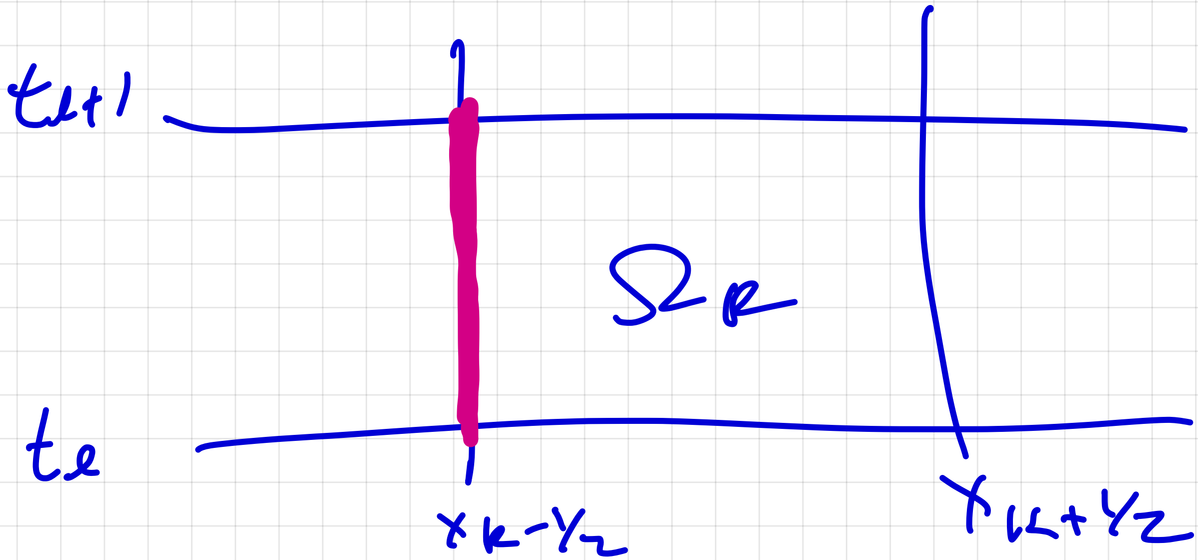
$\sigma = \dfrac{u_{k+1} - u_k}{h_x}$ Lax-Wendroff

$\sigma = \dfrac{u_k - u_{k-1}}{h_x}$ Beam-Warming

$\sigma = \dfrac{u_{k+1} - u_{k-1}}{2h}$ Fromm

Flux?

$$q(x, t_{\ell+1}) = q(x - a h_t, t_\ell)$$



$f^*_{k-1/2}$ = total flux across $x_{k-1/2}$ from t_ℓ to $t_{\ell+1}$

$$f_{k-1/2,\ell} = \frac{1}{h_t}\int_{t_\ell}^{t_{\ell+1}} a\, q(x_{k-1/2}, t)\, dt$$

q linear in Ω_{k-1}

$$= \frac{1}{h_x}\int_{t_\ell}^{t_{\ell+1}} a \cdot q\big(\underbrace{x_{k-1/2} - a(t-t_\ell)}, t_\ell\big)\, dt$$

$$= \frac{a}{h_t}\int_{t_\ell}^{t_{\ell+1}} u_{k-1\ell} + \delta_{k-1\ell}\big(x_{k-1/2} - a(t-t_\ell) - x_{k-1}\big)\, dt$$

$$= a\, u_{k-1} + \frac{a\,\delta_{k-1}}{h_t}\int_{t_\ell}^{t_{\ell+1}} \frac{h_x}{2} - a(t-t_\ell)\, dt$$

$$= a\, u_{k-1} + \frac{a\,\delta_{k-1}}{\cancel{h_t}}\left(\frac{\cancel{h_t}\, h_x}{2} - a\frac{h_t^{\cancel{2}}}{2}\right)$$

$$= a\, u_{k-1} + \frac{a}{2}(h_x - a h_t)\,\delta_{k-1,\ell}$$

next: δ_h has too much slope

look $r_k = \dfrac{u_k - u_{k-1}}{u_{k+1} - u_k}$