Today 3/8

1. Polynomial approximation

2. Norms, spaces, etc.

3. minimization or projection

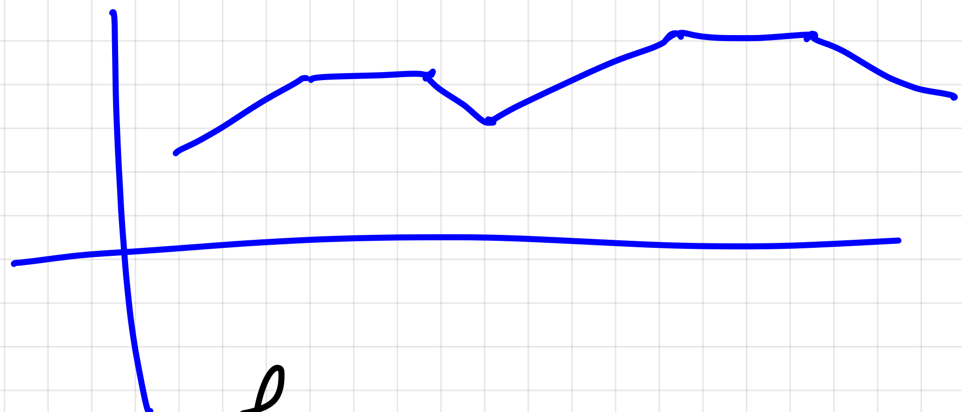
4. ... later, return to DG

Basic problem

→ Given $f(x)$ find a degree k polynomial $p_k(x)$ s.t.

$$p_k \sim f(x)$$

Q why do we want p_k?



another question

→ Given $Lu = f$ find u of degree k such that $f - Lu \sim 0$

We need a measure : $\rho_k \sim f(x)$

↑

a norm.

What is a norm?

A norm on a vector space V is a functional:

$$\|\cdot\| : V \to \mathbb{R}$$

$\cancel{\|f\| = \max |f|}$ -5

with 3 properties:

1. $\|f\| \geq 0$, $\|f\| = 0 \iff f = 0$
2. $\|\alpha f\| = |\alpha| \cdot \|f\| \quad \forall \alpha \in \mathbb{R}, f \in V$
3. $\|f + g\| \leq \|f\| + \|g\| \quad \forall f, g \in V$

What about a vector space?

A vector space (over $\mathbb{R}$) is a set that is closed under vector addition and scalar multiplication:

$\alpha \cdot f \in V$ if $\alpha \in \mathbb{R}$, $f \in V$

$f + g \in V$ if $f, g \in V$

Q Find a function space that is not a vector space.

$V = \{ v \mid v$ degree k on $[0,1]$, $v(0) = 0 \} \subset C([a,b])$

① max-norm:

$$\|f\|_\infty = \max_{a \le x \le b} |f(x)|$$

over $C([a,b])$

all continuous functions on closed $[a,b]$

② 2-norm:

$$\|f\|_2^2 = \int_a^b |f|^2 \, dx$$

over $C([a,b])$

let $P_k = \{ p(x) \mid p(x)$ is degree k poly. on $[a,b] \}$

min-max: find $p^*(x) = \underset{p(x)}{\operatorname{argmin}} \| f - p(x) \|_\infty$

least-squares: find $p^*(x) = \underset{p(x)}{\operatorname{argmin}} \| f - p(x) \|_2$

How?

Any $p_k(x)$ can be written

$$p_k(x) = \sum_{i=0}^{k} a_i \, \phi_i(x)$$

some polynomial basis

examples

$\phi_i = 1, x, x^2, x^3, \cdots x^k$ Monomials

$\phi_i = 1, x, \frac{3x^2-1}{2}, \frac{5x^3-x}{2}, \cdots$ Legendre

$$\phi_i = \frac{\prod_{j \neq i} (x - x_j)}{\prod_{j \neq i} (x_i - x_j)}$$ Lagrange

$$p_k(x) = \sum_{i=0}^{k} a_i \phi_i(x)$$

let $F(\underline{a}) = \| f(x) - p_k(x) \|_2^2$

$$\underline{a} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_k \end{bmatrix}$$

$$= \int_0^1 (f(x) - p_k(x))^2 \, dx$$

$$= \int_0^1 \left(f(x) - \sum a_i \phi_i(x) \right)^2 dx$$

$$= \int_0^1 f(x)^2 - 2\int_0^1 \sum a_i f \phi_i \, dx + \int_0^1 \sum_i \sum_j a_i a_j \phi_i \phi_j$$

$$\frac{\partial F}{\partial a_i} = 0 - 2 \sum_i a_i \int_0^1 f \phi_i \, dx$$

$$+ 2 \sum_i a_i \sum_j \int_0^1 a_j \phi_j \phi_i \, dx \overset{\text{set}}{=} 0$$

Solve for $\underline{a}$:

$$\sum_j a_j \underbrace{\int_0^1 \phi_j \phi_i \, dx}_{M_{ij}} = \int_0^1 f \phi_i \, dx \quad \text{for all } i$$

$$= \langle \phi_j, \phi_i \rangle$$

$$\underbrace{\begin{bmatrix} \langle \phi_0, \phi_0 \rangle & \cdots & \langle \phi_0, \phi_k \rangle \\ \vdots & & \vdots \\ \langle \phi_k, \phi_0 \rangle & \cdots & \langle \phi_k, \phi_k \rangle \end{bmatrix}}_{\substack{\text{Mass matrix} \\ M}} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_k \end{bmatrix} = \begin{bmatrix} \langle f, \phi_0 \rangle \\ \langle f, \phi_1 \rangle \\ \vdots \\ \langle f, \phi_k \rangle \end{bmatrix}$$

coefficients