Today 3/27

finite elements

- Review on FE so far
- FE weak forms
- FE assembly in 1D

$$\begin{cases} -u_{xx} = f(x) \\ u(0) = 0 \\ u'(1) = 0 \end{cases}$$

Dirichlet

Neumann

$$\int_0^1 -u_{xx} \cdot v \, dx = \int_0^1 f \, v \, dx$$

integration by parts

$$\int_0^1 u_x v_x \, dx - u_x v \Big|_0^1 = \int_0^1 f v \, dx$$

Find $u \in V$ ← some space such that

$$\int_0^1 u_x v_x \, dx - u_x v \Big|_0^1 = \int_0^1 f v \, dx$$

for all $v \in V$

for now

let $V = \{ u \mid \|u\| < \infty, \ u(0) = 0 \}$

→ find $u \in V$ s.t.

$$\int_0^1 u_x v_x \, dx \; - \cancel{u_x v \Big|_{x=1}} = \int_0^1 f v \, dx \quad \forall v \in V.$$

→ this imposes $u_x = 0$ at $x = 1$

→ find $u \in V$ s.t.

$$\underbrace{\underbrace{\int_0^1 u_x v_x \, dx}_{\langle u_x, v_x \rangle}}_{a(u,v)} = \underbrace{\int_0^1 f v \, dx}_{\langle f, v \rangle} \quad \forall v \in V.$$

↑ inner product.

bilinear form $\rightarrow$ $a(u,v) = \langle u_x, v_x \rangle$

linear functional $\rightarrow$ $F(v) = \langle f, v \rangle$

Find $u \in V$ s.t.

$$a(u,v) = F(v)$$

$\forall\, v \in V$.

let $V = \{ v \in L^2([0,1]) \mid a(v,v) < \infty,\ v(0) = 0 \}$

$L^2 = \{ u \mid \int_0^1 u^2\,dx < \infty \}$

Things we know

① Weak solutions are strong solutions

if $u \in C^2 \cap V$ Then $\underbrace{-u_{xx} = f}_{\text{strong}}$
and $\underbrace{a(u,v) = F(v) \ \forall v}_{\text{weak}}$

② The Ritz-Galerkin approximation is

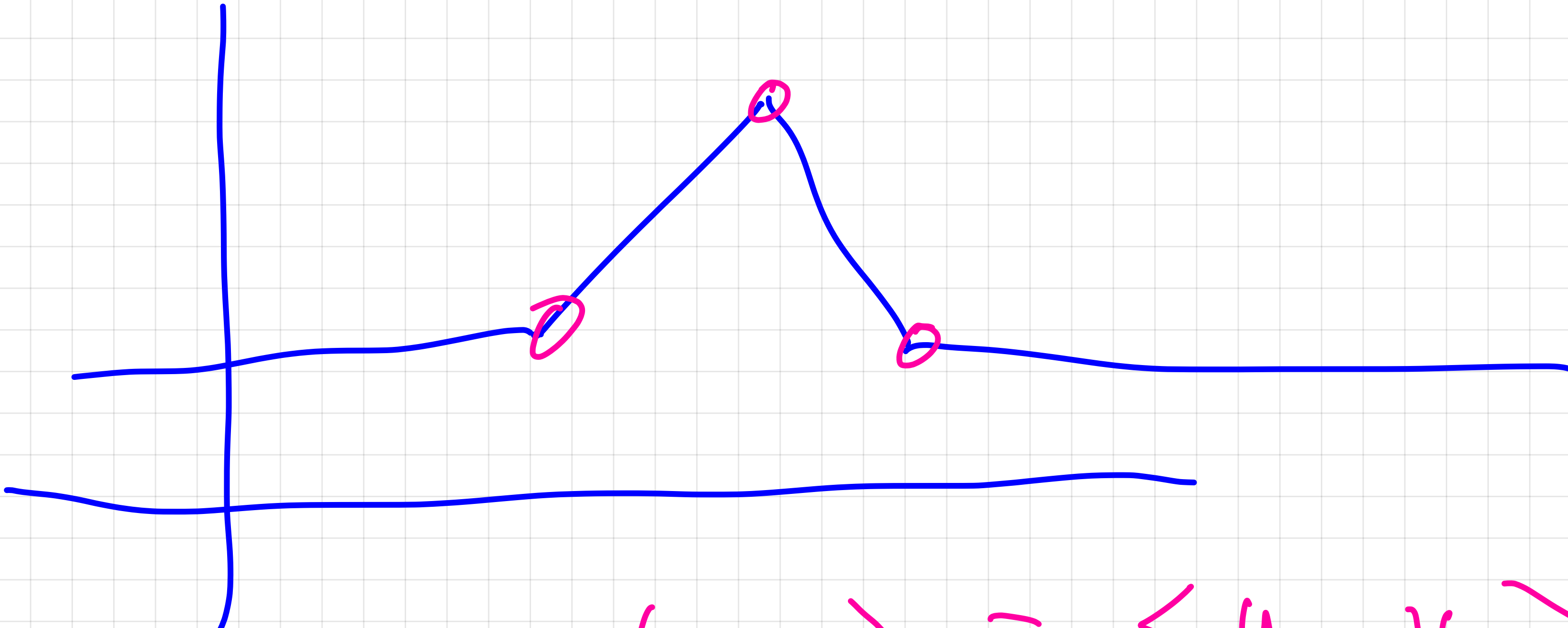
let $V^h \subset V$

find $u^h \in V^h$ s.t.

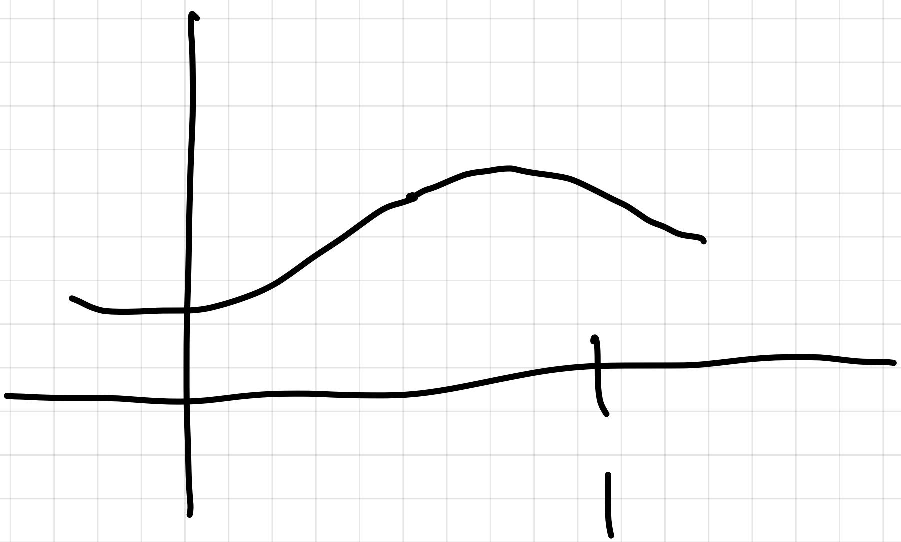
$a(u^h, v) = F(v) \ \forall v \in V^h.$

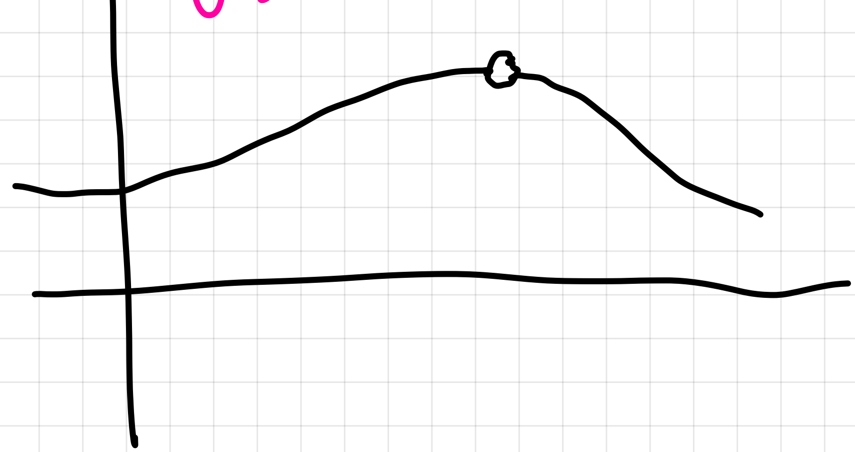
③ if $f \in L^2$

then there exists a unique $u^h$.



$$a(u,v) = \langle u_x , v_x \rangle = \int_0^1 u_x \, v_x \, dx$$





④ orthogonality:

if $u$ satisfies $a(u,v) = \langle f,v\rangle \quad \forall\, v \in V$

if $u^h$ satisfies $a(u^h,v^h) = \langle f,v^h\rangle \quad \forall\, v^h \in V^h$

then $e = u - u^h$ satisfies

$$a(e, v^h) = 0 \quad \forall\, v^h \in V^h.$$

⑤ $u^h$ is "the best" thing in $V^h$.

Ceá's Lemma

$$\|u - u^h\|_e = \min_{v^h \in V^h} \|u - v^h\|_e$$

$$\boxed{\|u\|_e^2 = a(u,u)}$$

strong: find $u \in C^2$ s.t. $-u_{xx} = f$

$u(0) = u'(1) = 0$

weak: find $u \in V$ s.t. $\langle u_x, v_x \rangle = \langle f, v \rangle$ $\forall v$.

discrete weak: find $u^h \in V^h \subset V$ s.t. $\langle u^h_x, v^h_x \rangle = \langle f, v^h \rangle$ $\forall v^h \in V^h$

$V = \{ v \mid \langle v_x, v_x \rangle < \infty$

$v(0) = 0 \}$

$V^h = ?$

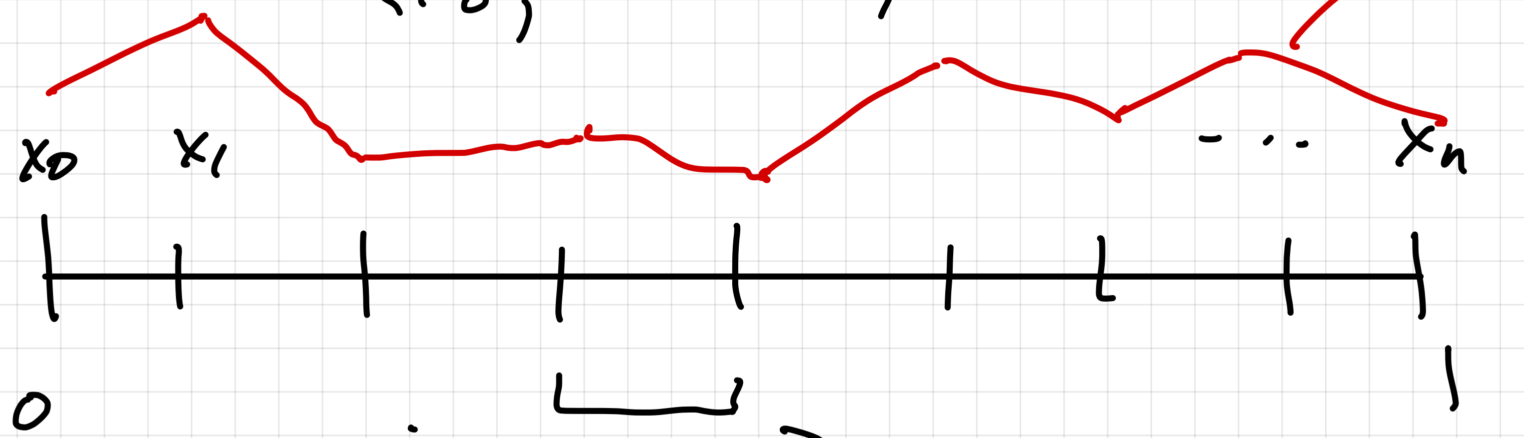
① cosines, sines?

② all polynomials of degree $\leq p$

③ all piecewise polynomials of degree $\leq p$

Define a mesh:

$$\{x_0, \dots, x_n\}$$

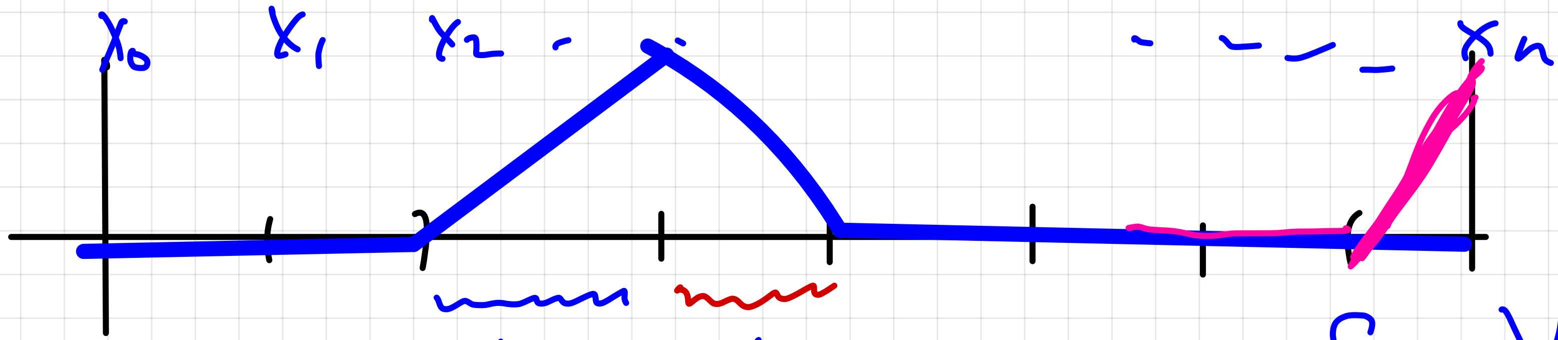



$$\tau^i = [x_{i-1}, x_i]$$

= "element" of the mesh.

let $V_k^h = \{ u \in C^0([0,1]) \mid u(x)$ is a polynomial of degree $\leq k$ on each $[x_{i-1}, x_i]\}$

k = degree



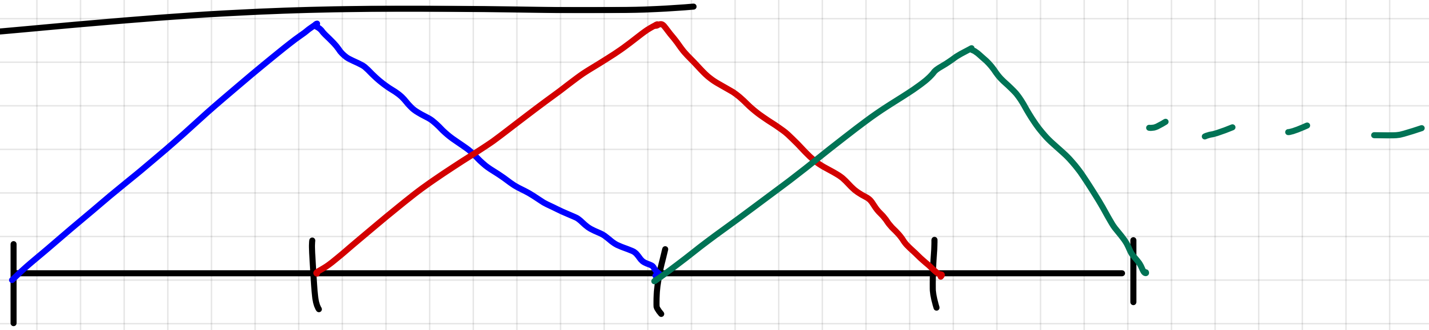

let $\{\phi_i\}$ be a basis for $V_1^h$

$$\phi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & x \in [x_{i-1}, x_i] \\[2ex] \dfrac{x_{i+1} - x}{x_{i+1} - x_i} & x \in [x_i, x_{i+1}] \\[2ex] 0 & \text{else} \end{cases}$$

$i = 1 \cdots n-1$

$$\phi_i(x) = \begin{cases} \dfrac{x - x_{n-1}}{x_n - x_{n-1}} \\[2ex] 0 \end{cases}$$

"The Interpolant"



let $w \in V$

the interpolant of $w$ is

$w_I^h \in V^h$.

$$w_I^h(x) = \sum_{i=1}^{n} w(x_i)\,\phi_i(x)$$

## Approximation Property (Thm 4.14)

let $h = \max_i h_i$

let $w \in C^2 \cap V$

Then $\| w - w_I^h \|_e \leq \frac{h}{\sqrt{2}} \| w'' \|$

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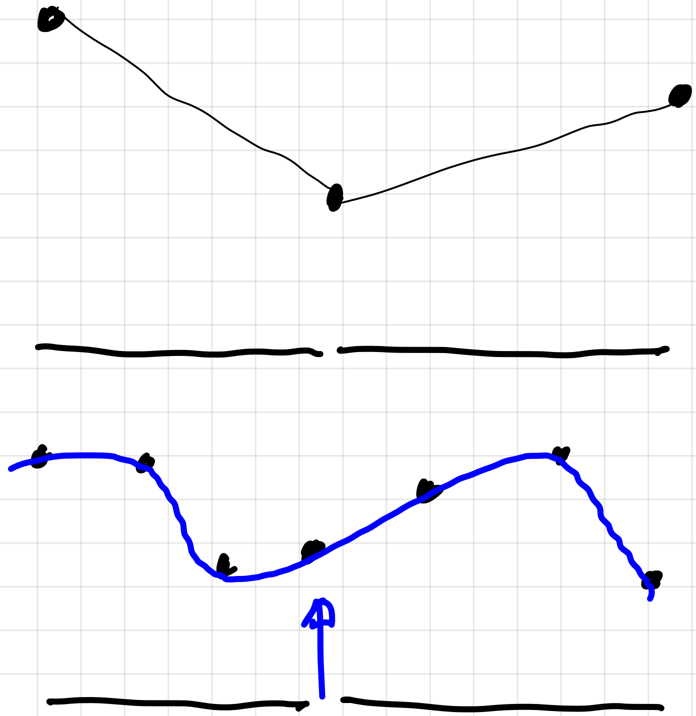
Result $f \in C^0$

$u \in C^2 \cap V$

then $\| u - u^h \| \leq \frac{h}{\sqrt{2}} \| u - u^h \|_e$

$\leq \frac{h^2}{2} \| f \|$

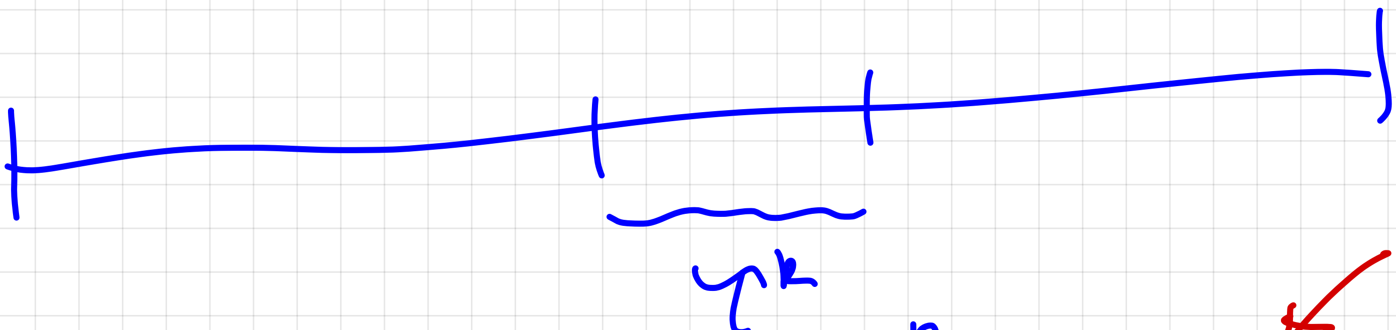


# Assembly

$$\langle u_x, v_x \rangle = \langle f, v \rangle \quad \forall v \in V^h$$

$\Omega = [0,1]$

$\tau^k = [x_{k-1}, x_k]$

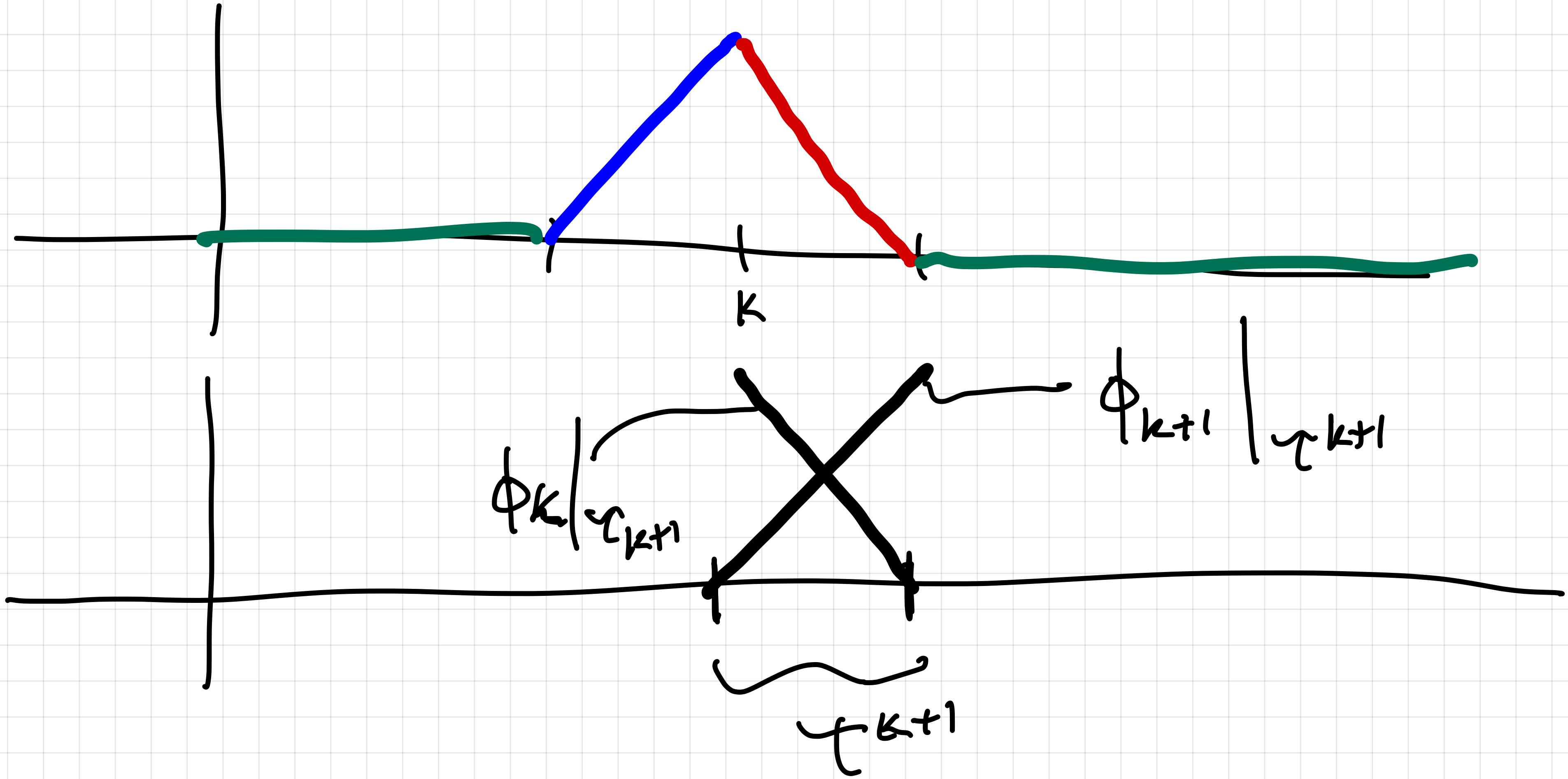



"hat" fcns

let $$u^h(x) = \sum_{j=1}^{n} u_j \phi_j(x)$$

→ Find $\underline{u}$ s.t.

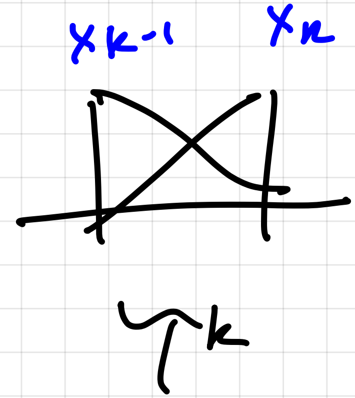
$$\langle u_h', \phi_i' \rangle = \langle f, \phi_i \rangle \quad \forall i = 1 \ldots n$$



k
φk | τk+1
φk+1 | τk+1
τk+1

let $\lambda_{k-1}(x) = \phi_{k-1}(x)\big|_{\tau^k}$

$$\lambda_k(x) = \phi_k(x)\big|_{\tau^k}$$




① linear ✓

② $\lambda_k + \lambda_{k-1} = 1$

$$\frac{x_k - x}{x_k - x_{k-1}} + \frac{x - x_{k-1}}{x_k - x_{k-1}} = 1$$

③ $$\frac{d\lambda_{k-1}}{dx} = \frac{1}{x_k - x_{k-1}}$$

$$\frac{d\lambda_k}{dx} = -\frac{1}{x_k - x_{k-1}}$$

$$\langle u_n', \phi_i' \rangle = \langle f, \phi_i \rangle_{i=1\cdots n}$$

$$u_n = \sum u_j \phi_j$$

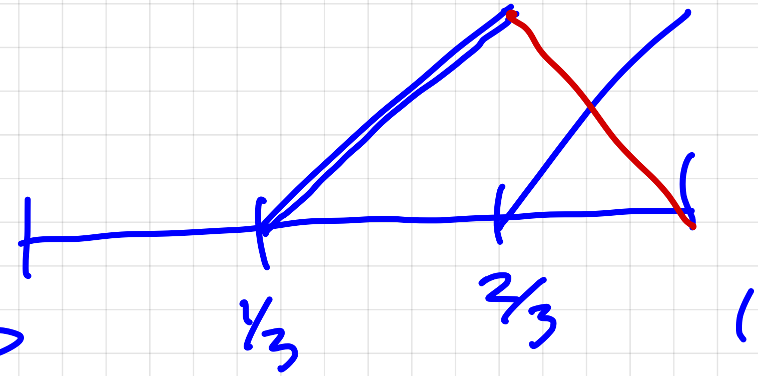
$$\sum_{j=1}^{n} u_j \underbrace{\int_0^1 \frac{d\phi_j}{dx} \cdot \frac{d\phi_i}{dx}\, dx}_{a_{ij}^{k}} = \int_0^1 f \phi_i \, dx$$

element stiffness matrix

$$\sum_{j=1}^{n} u_j \underbrace{\sum_{\tau^k \in T^h} \int_{\tau^k} \frac{d\phi_j}{dx} \frac{d\phi_i}{dx}\, dx}_{a_{ij}} = \sum_{\tau^k} \int_{\tau^k} f \phi_i \, dx$$

$$\hookrightarrow A \cdot \underline{u} = \underline{f}$$

## Example




$h = 1/3$

$$\int_{\tau_k} \phi_{k-1}' \, \phi_{k-1}' \, dx = \int_{\tau_k} \frac{1}{h} \cdot \frac{1}{h} = \frac{h}{h^2} = \frac{1}{h}$$

$$\int_{\tau_k} \phi_{k-1}' \, \phi_k' \, dx = \int_{\tau_k} \frac{1}{h} \left(-\frac{1}{h}\right) = -\frac{h}{h^2} = -\frac{1}{h}$$

$$A^k = \frac{1}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\langle f, \phi_i \rangle \qquad \text{let } f = 1$$

$$\rightarrow \int_{\tau_k} 1 \cdot \phi_i(x)\, dx = \int_{\tau} \frac{x - x_k}{h}$$

$$= \left. \frac{(x - x_k)^2}{2h} \right|_{\tau_k}$$

$$f^k = \frac{h}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A = \frac{1}{h}\begin{bmatrix} 1 & -1 & \\ -1 & 1 & \\ & & 0 \end{bmatrix} + \frac{1}{h}\begin{bmatrix} 0 & & \\ & 1 & -1 \\ & -1 & 1 \\ & & & 0 \end{bmatrix} = \frac{h}{2} + \frac{1}{h}\begin{bmatrix} 0 & & \\ & 1 & -1 \\ & -1 & 1 \end{bmatrix}$$

$$= \frac{1}{h}\begin{bmatrix} 1 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 1 \end{bmatrix} \qquad f = \frac{h}{2}\begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

$$u_0 = 0$$

$$\rightarrow \begin{bmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \frac{h}{2} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$