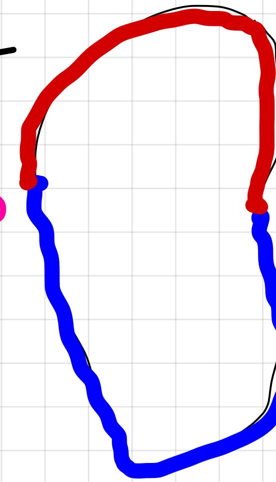
Today

1. 2D FE assembly (§ 8.4)
2. Project goals

Model Problem

$$-\nabla\cdot k(\underline{x})\nabla u = f \quad \text{in } \Omega$$



$$u = g_D$$

$$n\cdot k(x)\nabla u = n\cdot g_N$$

for now

Q What is the weak form of this problem?

$\cancel{\text{let } \underline{q} = \sqrt{k}\,\nabla u}$

$\cancel{-\nabla\cdot\sqrt{k}\,\underline{q} = f}$

1D

$$-u'' = f$$

$$\int_\Omega -u'' v\,dx = \int f v\,dx$$

$$\int_\Omega u'v'\,dx - u'v\Big|_0^1 = \int_\Omega f v\,dx$$

$$-\nabla \cdot k(x) \nabla u = f$$

$$\int_\Omega -\nabla \cdot k(x) \nabla u \; v \, dx = \int_\Omega f v \, dx$$

I.B.P.

$$\int_\Omega k(x) \nabla u \cdot \nabla v \, dx - \int_{\partial\Omega} n \cdot k(x) \nabla u \; v \, dx = \int_\Omega f v \, dx$$

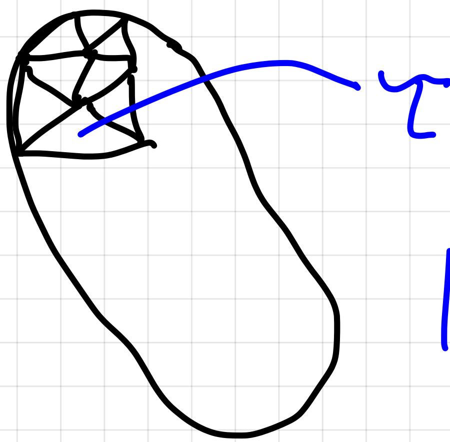
let V be a space with $u(\Gamma_D) = 0$

find $u \in V$ s.t.

$$\int_\Omega K \nabla u \cdot \nabla v \, dx - \underbrace{\int_{\Gamma_N} n \cdot K \nabla u \, v \, dx}_{\text{let} = 0} = \int_\Omega f v \, dx$$

find $v \in V$ s.t. What? (next week)

$\rightarrow$ $$\int_\Omega k \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \quad \forall v \in V$$



let $V^h = \{ v \in C^0(\Omega) \mid v = a + bx + cy = \text{linear on each } \tau \}$

$$\int_\Omega k(\underline{x}) \nabla u \cdot \nabla v \, d\underline{x}$$

Find $u^h \in V^h$ s.t.

$$a(u^h, v^h) = \langle f, v^h \rangle \quad \forall\, v^h \in V^h$$

$$\int_\Omega K(\underline{x})\, \nabla u \cdot \nabla v \, d\underline{x}$$

① Assembly

② Apply B.C

over the elements τ:

$$\sum_\tau \int_\tau K\, \nabla u \cdot \nabla v \, d\underline{x} = \sum_\tau \int_\tau f v$$

let $\{\phi_i(\underline{x})\}$ be a basis for V^h.

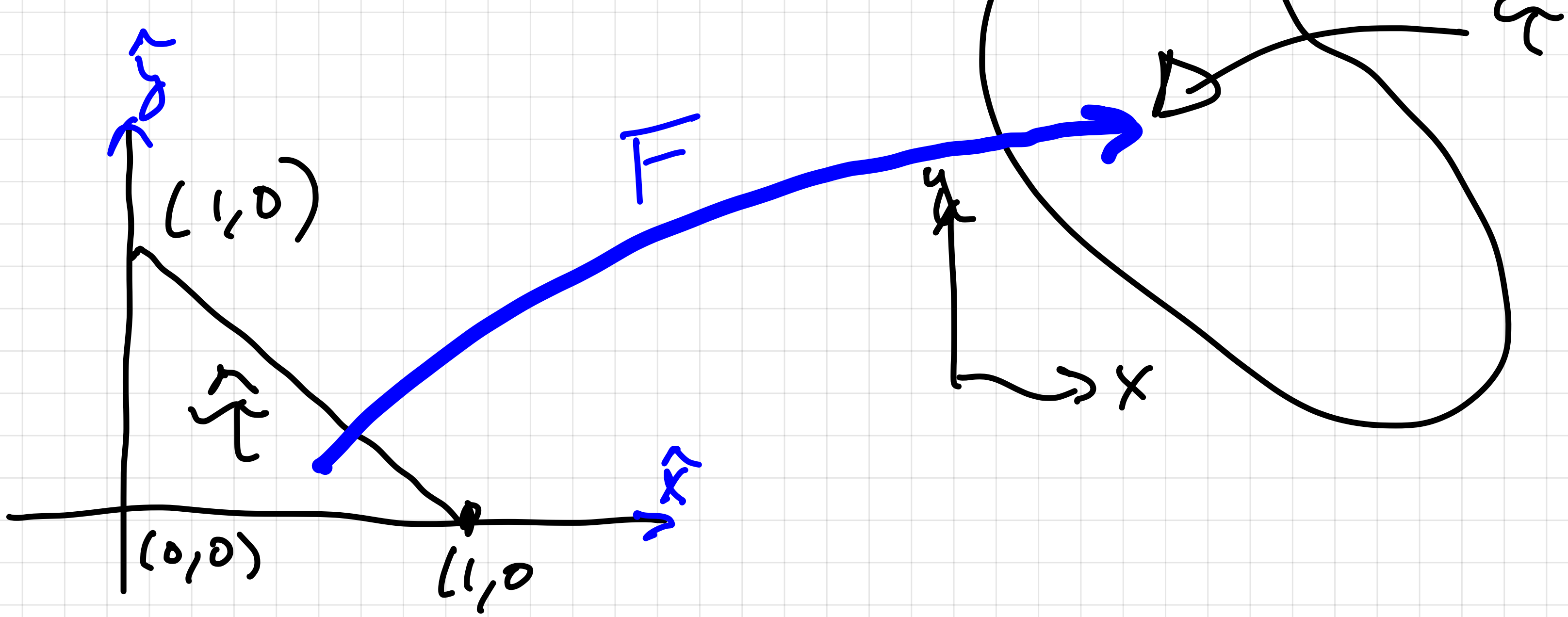
let $A_{ij} = a(\phi_j, \phi_i)$

$$= \sum_{\tau} \int_{\tau} K(\underline{x}) \nabla\phi_j \nabla\phi_i \, dx$$

$$F_i = \sum_{\tau} \int_{\tau} f(\underline{x}) \phi_i(x) \, dx$$

$$\underline{\underline{A}} \, \underline{u} = \underline{F}$$

Which bases?



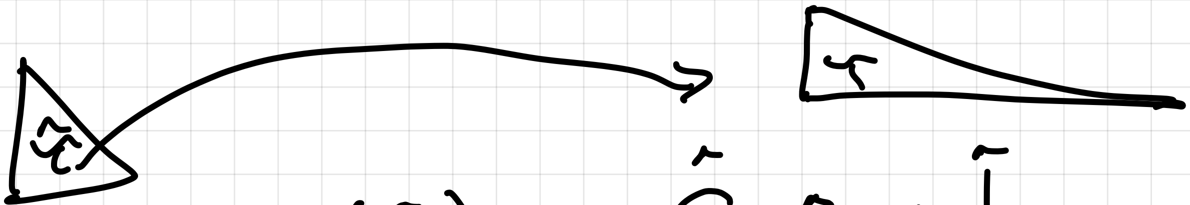
want any $a + bx + c$

try $\hat{\phi}_0(\hat{\underline{x}}) = 1 - \hat{x} - \hat{y}$

$\hat{\phi}_1(\hat{\underline{x}}) = \hat{x}$

$\hat{\phi}_2(\hat{\underline{x}}) = \hat{y}$

let $F : \hat{\tau} \longrightarrow \tau$



$$F(\hat{x}) = \hat{C}\,\underbrace{\hat{\underline{x}} + \bar{\underline{b}}}_{\text{affine}}$$

$$\rightarrow F = \begin{bmatrix} x_1 - x_0 & x_2 - x_0 \\ y_1 - y_0 & y_2 - y_0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

Jacobian $J_F = \begin{bmatrix} x_1 - x_0 & x_2 - x_0 \\ y_1 - y_0 & y_2 - y_0 \end{bmatrix}$

Have a basis on $\hat{\tau}$: $1 - \hat{x} - \hat{y}$

$\hat{x}$

$\hat{y}$

What is the basis on τ ?

let $\underline{x} \in \tau$

then $F^{-1}(\underline{x}) \in \hat{\tau}$

so $\phi_j := \hat{\phi}_j(F^{-1}(\underline{x}))$ is a linear in $\hat{\tau}$

Also $\nabla \phi_j = \nabla(\hat{\phi}_j(F^{-1}(\underline{x})))$

$= J_F^{-T} \nabla_{\hat{x}}(\hat{\phi}(F^{-1}(\underline{x})))$

why? chain rule.

$$\rightarrow \int_{\tau} k(\underline{x})\, \nabla\phi_j \cdot \nabla\phi_i \, dx = \int_{\hat{\tau}} k(F(\hat{x}))\, \underbrace{(J_F^{-T} \nabla_{\hat{x}} \hat{\phi}_j) \cdot (J_F^{-T} \nabla_{\hat{x}} \hat{\phi}_i)}\, |J_F|\, d\hat{\underline{x}}$$

$$\rightarrow \int_{\tau} f(\underline{x})\, \phi_i(\underline{x})\, dx = \int_{\hat{\tau}} f(F(\hat{x}))\, \hat{\phi}_i \, |J_F|\, d\underline{x}$$

$$\hat{\phi} = 1 - \hat{x} - \hat{y} \rightarrow \nabla_{\hat{x}} \hat{\phi} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$\hat{\phi} = \hat{x} \rightarrow \nabla_{\hat{x}} \hat{\phi} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

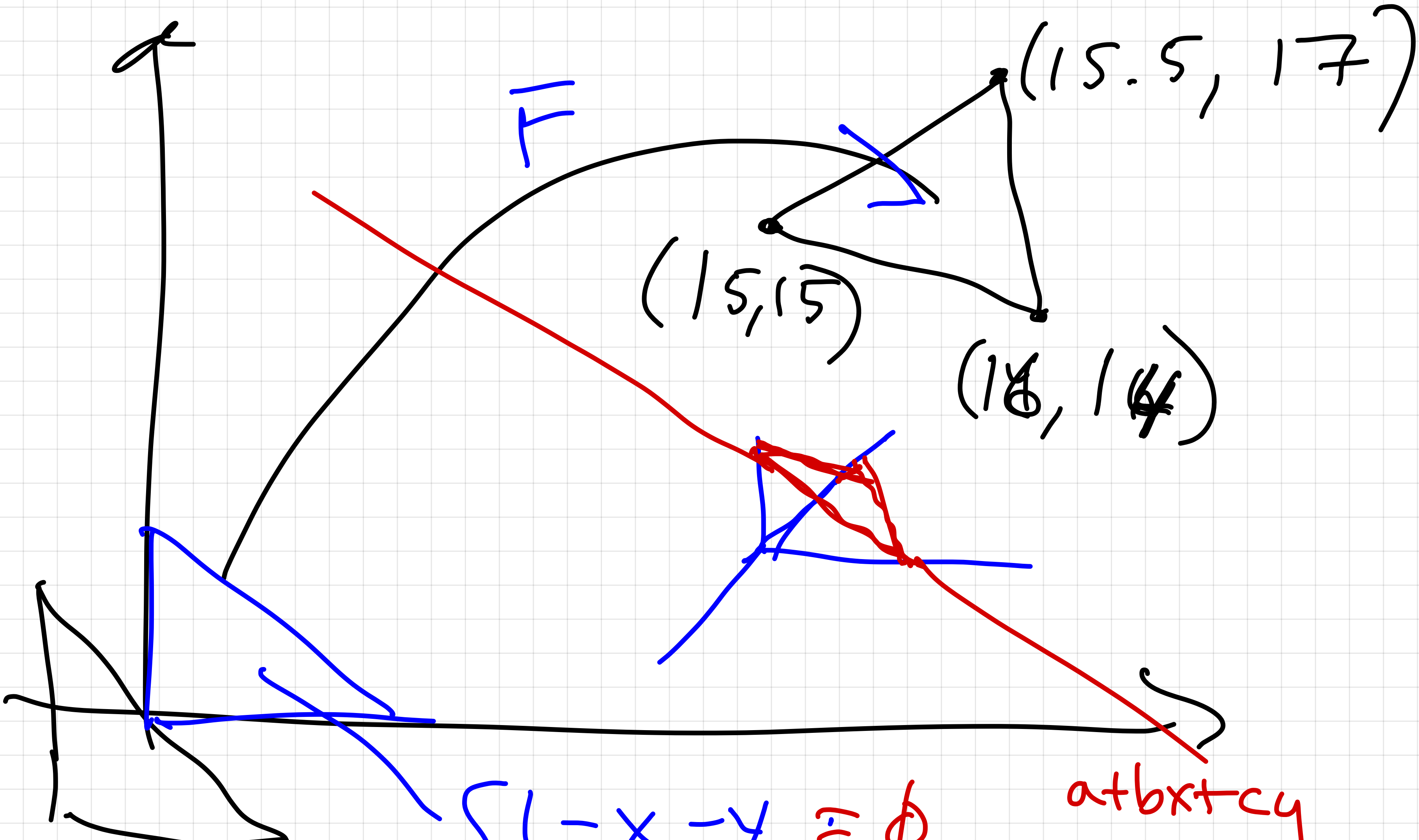
$$\hat{\phi} = \hat{y} \rightarrow \nabla_{\hat{x}} \hat{\phi} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

find $u^h = \sum u_j \phi_j$ s.t.

$$\int k \nabla u^h \cdot \nabla \phi_i = \int f \phi_i \quad \forall \phi_i$$

for each τ:

$$\left.\begin{array}{l} \displaystyle\int_\tau k \nabla\phi_0 \cdot \nabla\phi_0 \\ \displaystyle\int k \nabla\phi_0 \nabla\phi_1 \\ \displaystyle\int k \nabla\phi_0 \nabla\phi_2 \\ \displaystyle\int k \nabla\phi_1 \nabla\phi_0 \\ \quad\quad\quad\vdots \end{array}\right\} 9$$



$$\begin{cases} 1 - x - y = \phi_0 \\ x = \phi_1 \\ y = \phi_2 \end{cases}$$

$$a + bx + cy = \sum u_i \phi_i$$