Today 1/24

Objectives

1. Outline expectations for $u_t + a u_x = 0$

2. Describe ETBS as a 2-level scheme.

3. Introduce convergence
   consistency
   stability

$$\begin{cases} u_t + a\,u_x = 0 \\ u(0) = u(1) \end{cases}$$

$a > 0$

periodic

→ this is "advection"

→ $$\frac{\partial u(x,t)}{\partial t} + a\,\frac{\partial u(x,t)}{\partial x} = 0$$

What happens to $u(x,t)$ along some curve $x(t)$?

→ $$\frac{d}{dt}\,u(x(t),t) = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial t}$$

if $\frac{dx}{dt} = a$ then $$\frac{\partial u}{\partial x}\,a + \frac{\partial u}{\partial t} = u_t + a\,u_x = 0$$

⇒ $x(t) = x_0 + a\,t$. $u(x,t)$ is constant along these curves

$$u_t + a u_x = 0$$

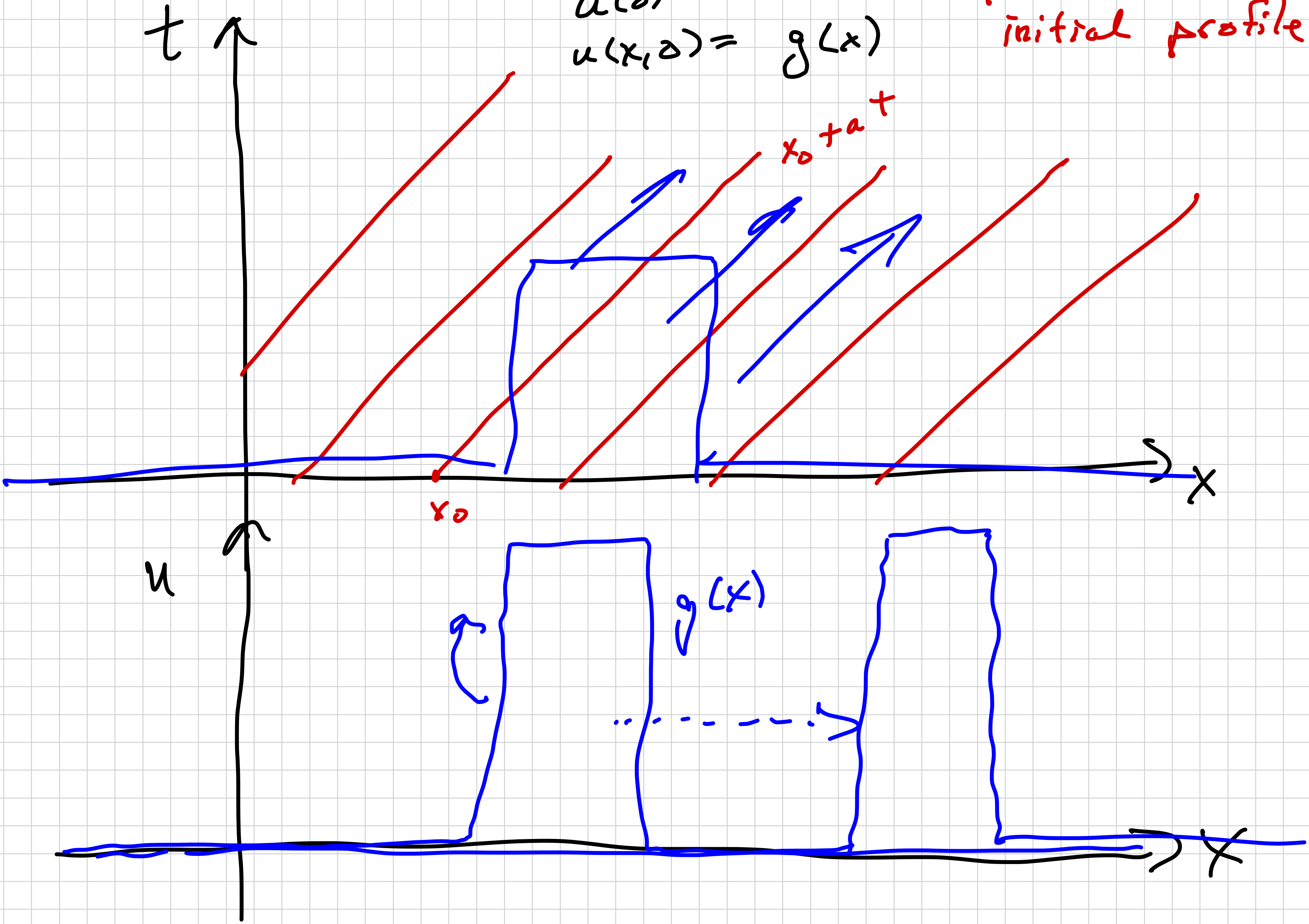
$$u(0) = u(1)$$

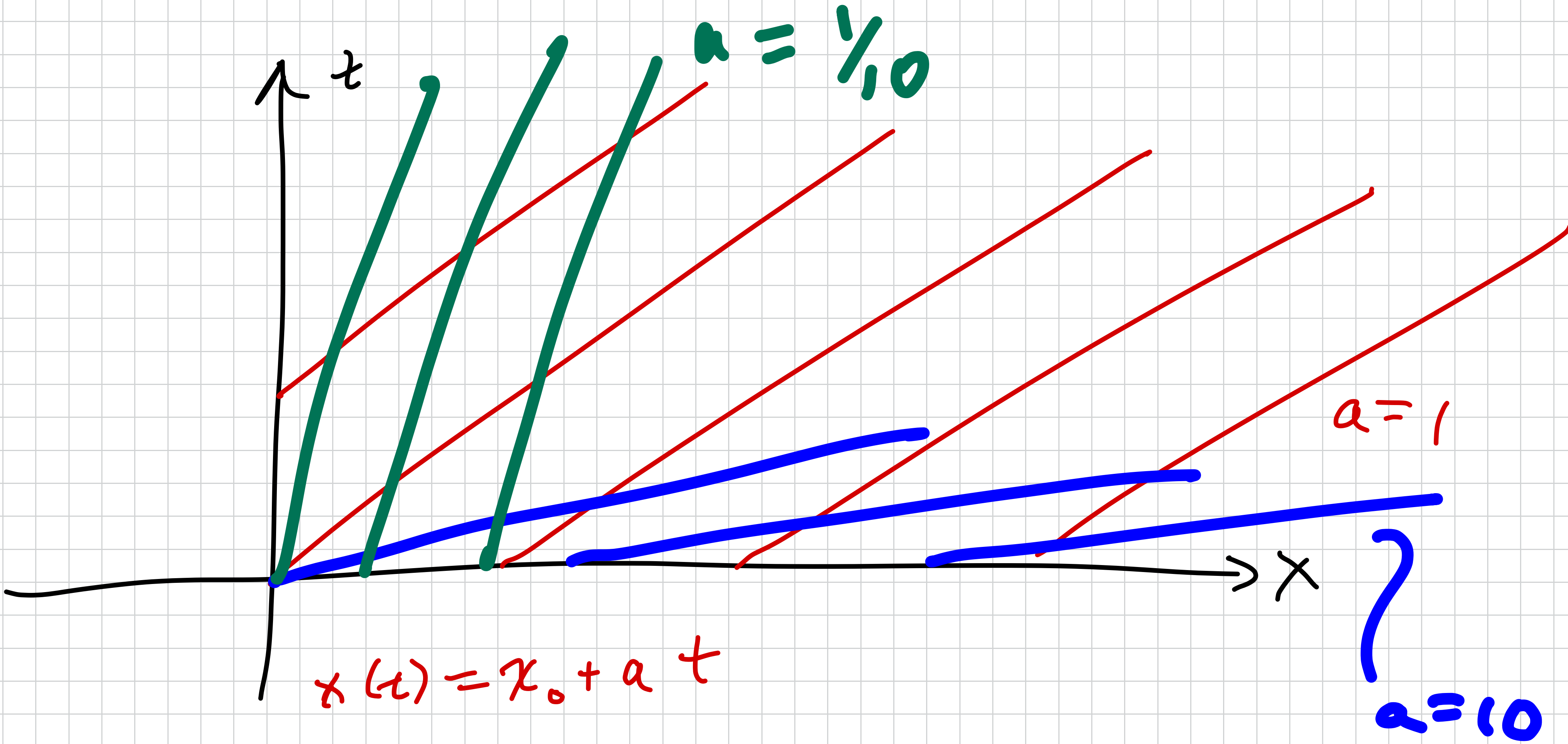
$$u(x,0) = g(x)$$

$a > 0$

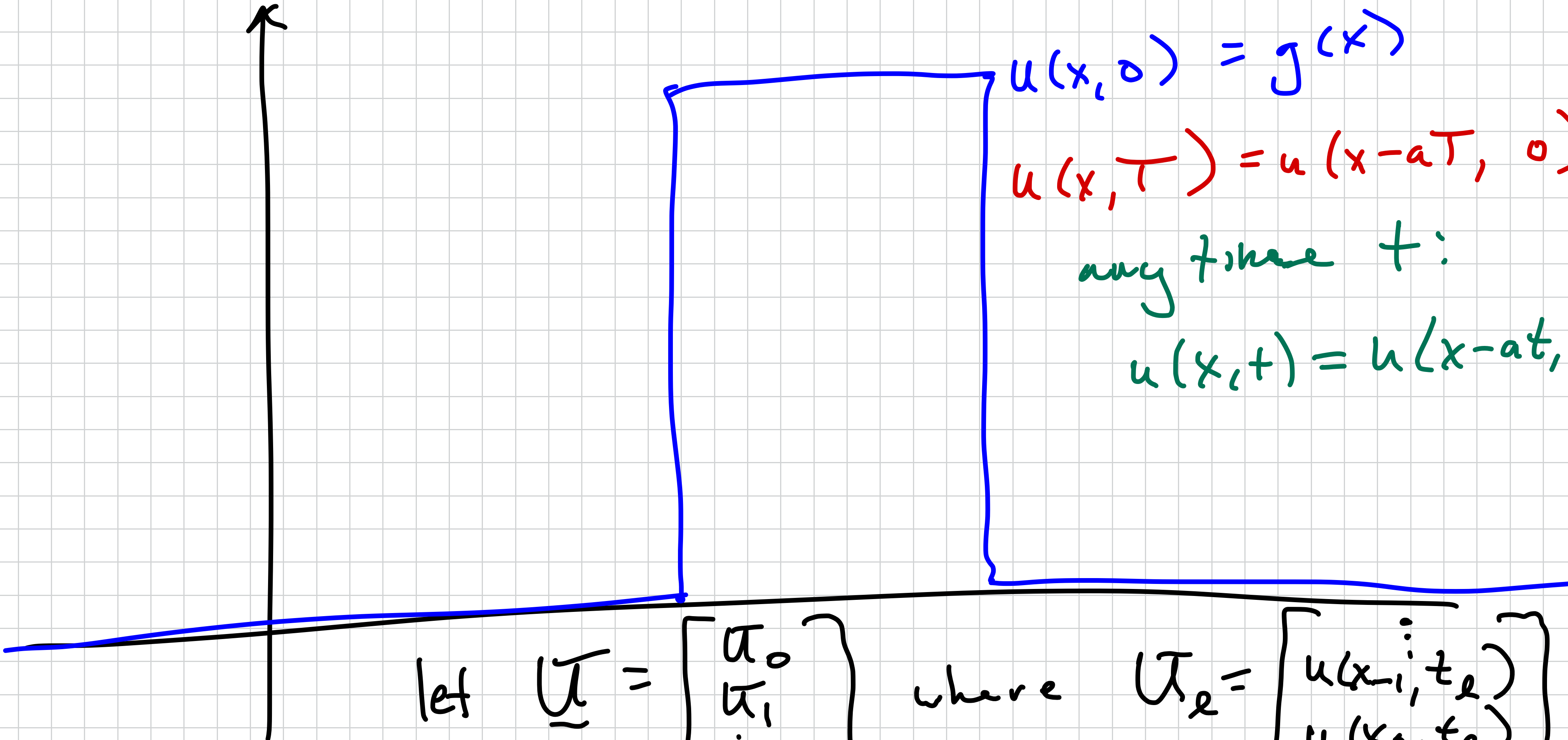
periodic

initial profile





t
x
a = 1/10
a = 1
a = 10
x(t) = x₀ + a t



$u(x,0) = g(x)$

$u(x,T) = u(x-aT,\ 0)$

any time t:

$u(x,t) = u(x-at, 0)$

let $\underset{\sim}{U} = \begin{bmatrix} U_0 \\ U_1 \\ \vdots \end{bmatrix}$ where $U_\ell = \begin{bmatrix} \vdots \\ u(x_{-1}, t_\ell) \\ u(x_0, t_\ell) \\ u(x_1, t_\ell) \\ \vdots \end{bmatrix}$

= exact solution to $u_t + a u_x = 0$

let $\underline{u} = \begin{bmatrix} \underline{u}_0 \\ \underline{u}_1 \\ \vdots \end{bmatrix}$ where $u_\ell = \begin{bmatrix} \vdots \\ u_{k-1,\ell} \\ u_{k,\ell} \\ u_{k+1,\ell} \\ \vdots \end{bmatrix}$

= approximation

Then the error $e_{k,\ell} = u(x_k, t_\ell) - u_{k,\ell}$

or

$$\underline{e} = \underline{U} - \underline{u} \quad \text{or} \quad \underline{e}_\ell = \underline{U}_\ell - \underline{u}_\ell$$

Q: How do we know if our scheme is <u>accurate</u>?

$\Rightarrow$ $e \to 0$ as $h_t, h_x \to 0$

$\Rightarrow$ "convergent"

Definition 5.7: Two-Level Linear Finite-Difference Scheme

A finite-difference scheme that can be written as,

$$P_h \boldsymbol{u}_{\ell+1} = Q_h \boldsymbol{u}_\ell + \cancel{h\boldsymbol{b}_\ell}, \tag{5.5}$$

is called a two-level linear finite-difference scheme. Each iteration depends only on two instances of time. Examples are given in Example 5.8.

ETBS

$$\frac{u_{k\ell+1} - u_{k\ell}}{h_t} + a\,\frac{u_{k\ell} - u_{k-1\ell}}{h_x} = 0$$

$$u_{k\ell+1} - u_{k\ell} + \underbrace{\frac{a h_t}{h_x}}_{\lambda}\left(u_{k\ell} - u_{k-1\ell}\right) = 0$$

$$1 \quad u_{k\ell+1} = \lambda\, u_{k-1\ell} + (1-\lambda)\, u_{k\ell}$$

$\Rightarrow P = I$

$Q = \text{tridiag}(\lambda,\ 1-\lambda,\ 0)$

$$\mathrm{tridiag}(\lambda, 1-\lambda, 0) = \begin{bmatrix} 1-\lambda & 0 & & \\ \lambda & \ddots & \ddots & \\ & \lambda & 1-\lambda & 0 \\ & & \ddots & \ddots \end{bmatrix} \cdot \begin{bmatrix} u_{k-1\,l} \\ u_{k\,l} \\ u_{k+1\,l} \end{bmatrix}$$

Truncation error:

Definition 5.10: Truncation Error

The local truncation error, $\tau_{k,\ell}$, is the error that remains when a finite-difference method is applied to the exact solution, $u(x_k, t_\ell)$.

Example 5.12: ETFS Truncation Error

$$
\begin{aligned}
\tau_{k,\ell} =& \frac{u(x_k, t_{\ell+1}) - u(x_k, t_\ell)}{h_t} + a\frac{u(x_{k+1}, t_\ell) - u(x_k, t_\ell)}{h_x} \\
=& \frac{1}{h_t}\left(u(x_k, t_\ell) + u_t(x_k, t_\ell)h_t + u_{tt}(x_k, \varsigma)\frac{h_t^2}{2} - u(x_k, t_\ell)\right) \\
&+ \frac{a}{h_x}\left(u(x_k, t_\ell) + u_x(x_k, t_\ell)h_x + u_{xx}(\xi^+, t_\ell)\frac{h_x^2}{2} - u(x_k, t_\ell)\right) \\
=& u_{tt}(x_k, \varsigma)\frac{h_t}{2} + a u_{xx}(\xi^+, t_\ell)\frac{h_x}{2} \\
=& \mathcal{O}(h_t, h_x)
\end{aligned}
$$

Taylor expansion in time

Taylor exp in space

$\tau = O(h_t, h_x)$

Definition 5.15: Consistency, Stability, and Convergence

Let $\frac{\partial^\mu u}{\partial x^\mu}$ denote the μ-th partial derivative in x, and $\frac{\partial^\nu u}{\partial t^\nu}$ denote the ν-th partial derivative in t. Assume that $\frac{\partial^\mu u(x,\hat{t})}{\partial x^\mu}, \frac{\partial^\nu u(x,\hat{t})}{\partial t^\nu} \in L^2(\mathbb{R})$, for all $\hat{t} < t^*$.

A two-level linear finite-difference scheme, $P_h \boldsymbol{u}_{\ell+1} = Q_h \boldsymbol{u}_\ell + \mathbf{b}_\ell h_t$, is

- *consistent in the L^2-norm with order ν in time and μ in space if*

$$\max_{\ell, \ell h_t \le t^*} \|\boldsymbol{\tau}_\ell\| = \mathcal{O}(h_x^\mu, h_t^\nu);$$

- *convergent in the L^2-norm with order ν in time and μ in space if*

$$\max_{\ell, \ell h_t \le t^*} \|\boldsymbol{e}_\ell\| = \mathcal{O}(h_x^\mu, h_t^\nu);$$

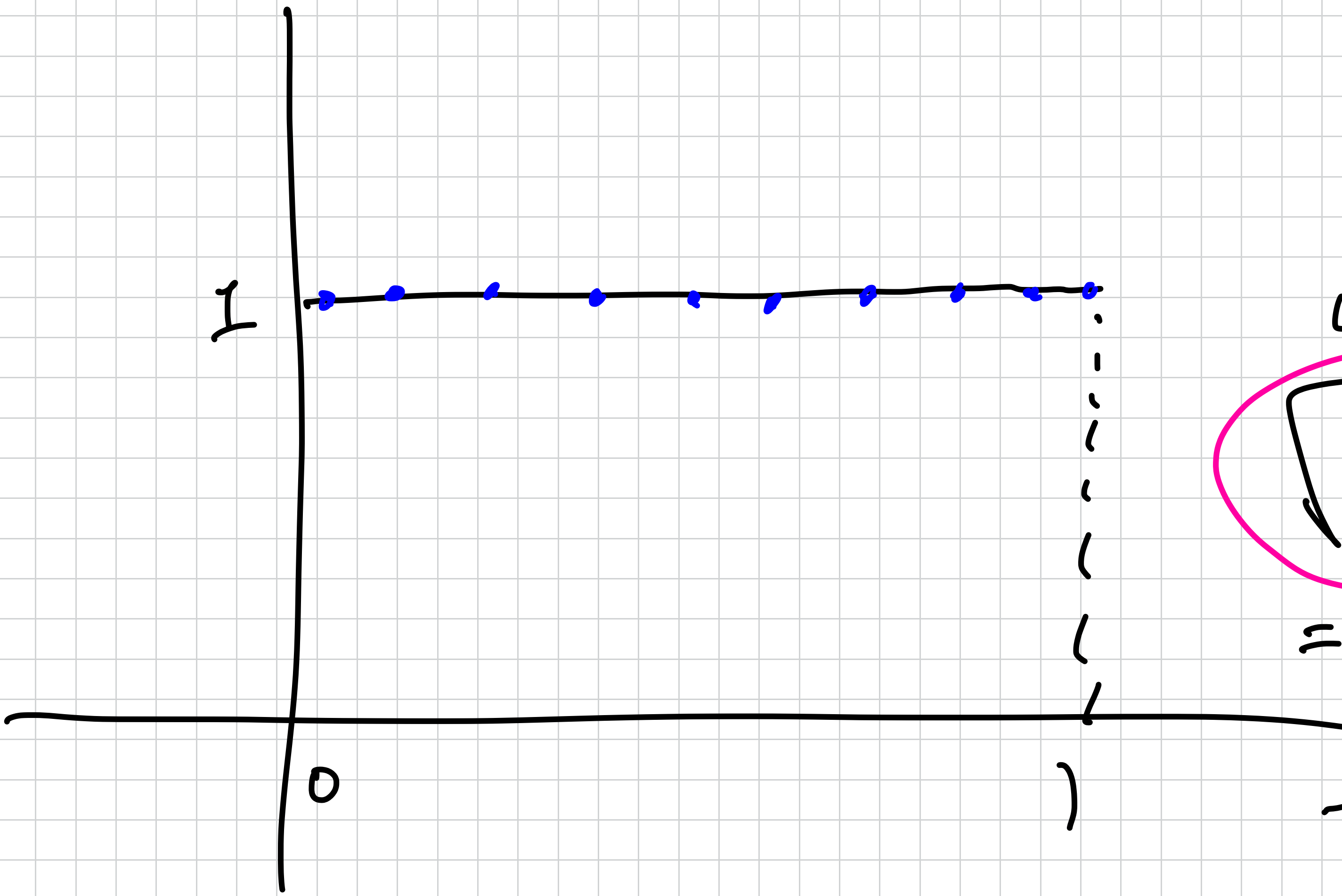
- *stable in the L^2-norm if $\exists c > 0$, independent of h_t and h_x, such that $\|(P_h^{-1} Q_h)^\ell P_h^{-1}\| \le c$ for all ℓ and h_t such that $\ell h_t \le t^*$.*

$P u_{\ell+1} = Q u_\ell$

$u_0 \to u_1 = P^{-1} Q u_0$

$u_2 = P^{-1} Q u_1 = (P^{-1} Q)^2 u_0 \to u_3 = P^{-1} Q u_2$

$= (P^{-1} Q)^3 u_0$



$$u(x) = 1$$

$$\sqrt{\int_0^1 u^2\,dx}$$

$$= \sqrt{\int_0^1 1\,dx}$$

$$= \sqrt{1}$$

$$= 1$$

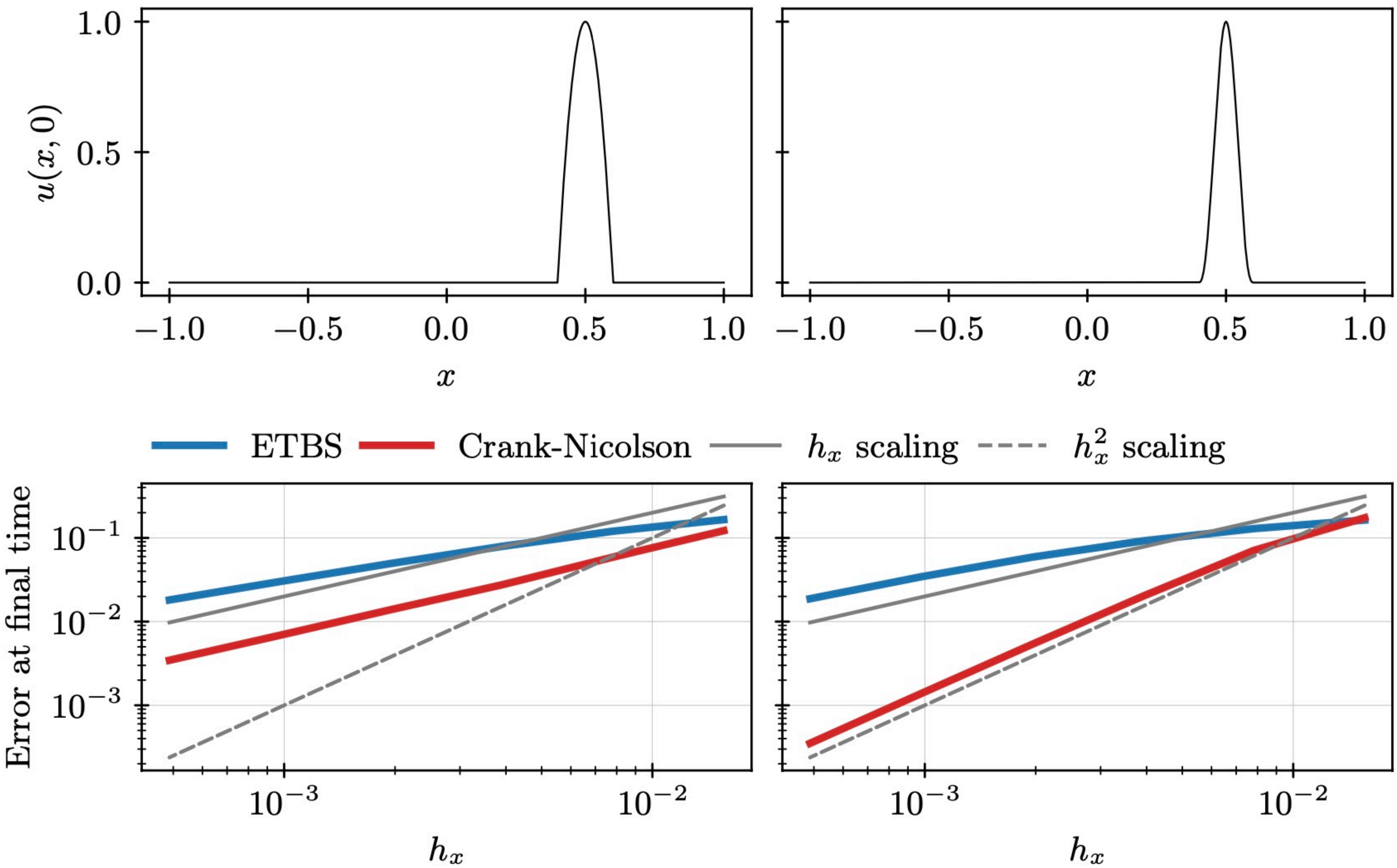
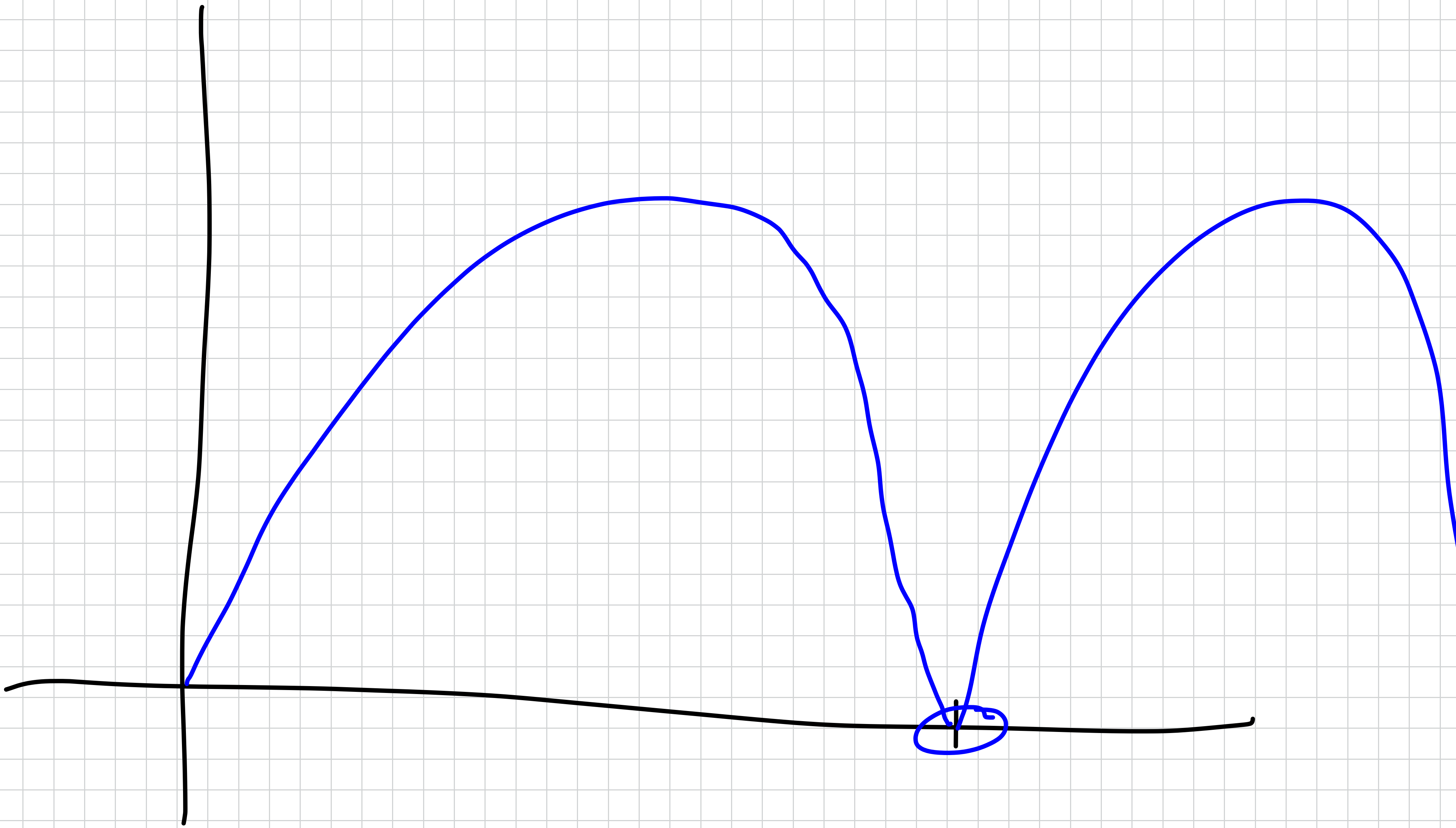


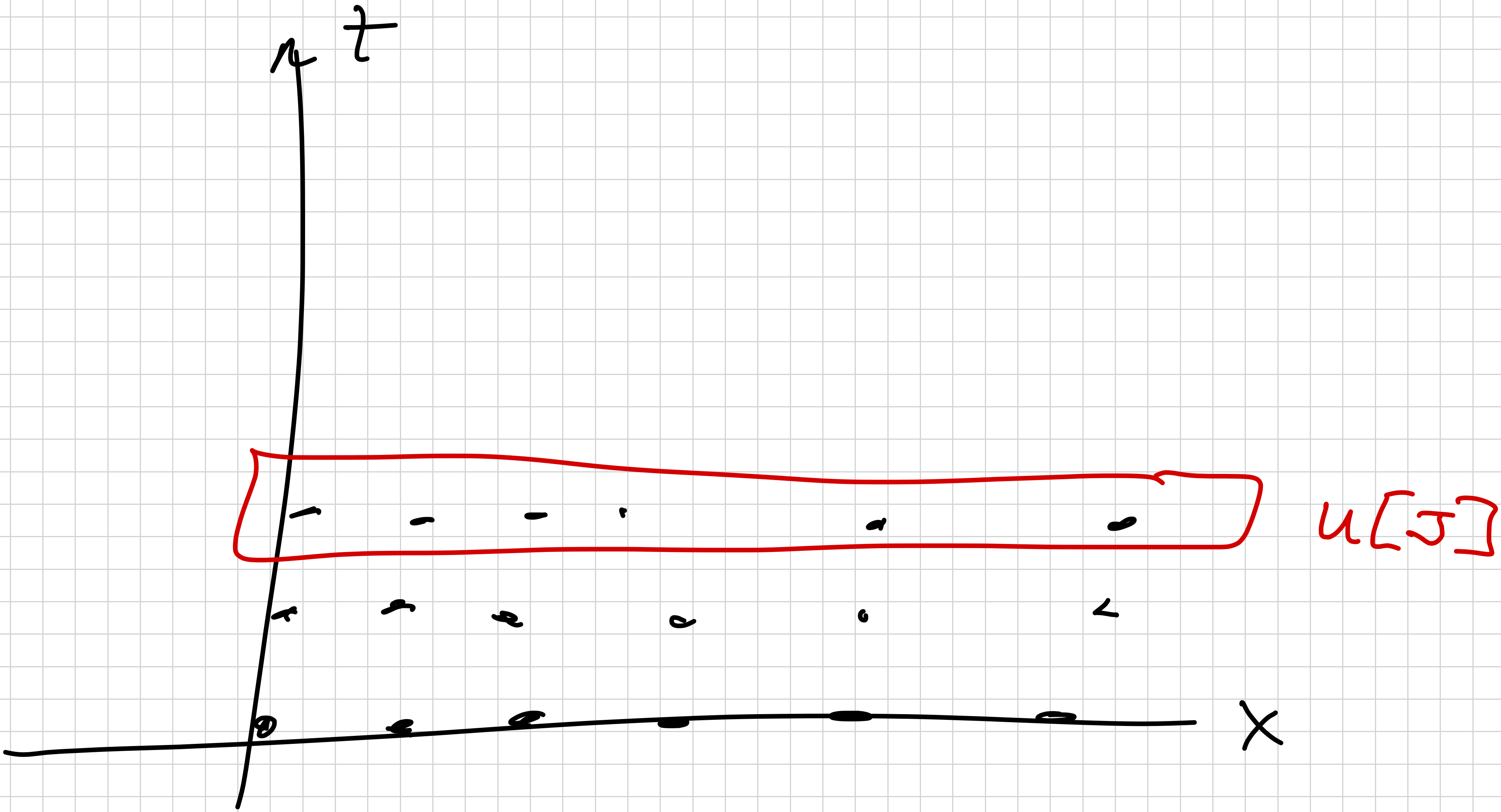
Figure 5.2.1. *Convergence of two schemes for the one-way wave equation, with a continuous, but not continuously differentiable solution (left), and a smoother solution (right).*



Theorem 5.19: Lax Convergence Theorem

If a two-level linear finite-difference scheme, $P_h \boldsymbol{u}_{\ell+1} = Q_h \boldsymbol{u}_\ell + \mathbf{b}_\ell h_t$, *is* consistent *in the* L^2*-norm with order* ν *in time and* μ *in space* ***and*** stable *in the* L^2*-norm, then it is* convergent *in the* L^2*-norm with order* ν *in time and* μ *in space.*

$$\sqrt{\sum h_x h_y (\cdot)^2}$$


t
X
u[J]