Today 1/29

Topic: stability

Objectives

1. Set up stability in terms of matrices

2. Derive von Neumann stability.

Two level schemes

$$P\, u_{\ell+1} = Q\, u_\ell \quad + \cancel{h_t b_\ell}$$

ignore

$\underline{u}$ = numerical approx.

$\underline{U}$ = exact.

$\underline{e} = \underline{U} - \underline{u}$

The truncation error $\underline{\tau}$ is the remainder after apply the FD scheme to the exact solution.

How do errors propagate?

$$P\,U_{\ell+1} = Q\,U_\ell + \tau\,h_t$$

$$P\,u_{\ell+1} = Q\,u_\ell$$

$$\Rightarrow P\,e_{\ell+1} = Q_\ell + \tau h_t$$

$$e_0 = 0 \quad (\text{since } u_0 = U_0)$$

$= \text{initial cond.}$

$$P e_1 = Q e_0 + \tau_0 h_t$$
$$= \tau_0 h_t$$
$$e_1 = P^{-1} \tau_0 h_t$$
$$P e_2 = Q e_1 + \tau_1 h_t$$
$$\Rightarrow e_2 = P^{-1} Q\, e_1 + P^{-1} \tau_1 h_t$$

$$e_2 = P^{-1} Q P^{-1} \tau_0 h_t + P^{-1} \tau h_t$$

$\vdots$

$$P e_3 = Q e_2 + \tau_2 h_t \Rightarrow e_3 = P^{-1} Q e_2 + P^{-1} \tau h_t$$

$$= (P^{-1} Q)^2 P^{-1} \tau_0 + P^{-1} \tau_2 h_t$$

$\vdots$

$$e_4 =$$

$\vdots$

$$e_\ell = \sum_{m=1}^{\ell} (P^{-1} Q)^{\ell - m} P^{-1} \tau_0 h_t$$

$$+ P^{-1} \tau_0 h_t$$

want

$$\Rightarrow \|e_\ell\| \leq h_t \sum_{m=1}^{\ell} \underbrace{\|(P^{-1} Q)^{\ell-m} P^{-1}\|}_{} \underbrace{\|\tau\|}_{\text{consistency}}$$

$$P u_{\ell+1} = Q u_\ell$$

$$P \bar{u}_{\ell+1} = Q \bar{u}_\ell + \tau_\ell h_t$$

$$\Rightarrow P e_{\ell+1} = Q e_\ell + \tau_\ell h_t$$

$$\Rightarrow e_{\ell+1} = P^{-1} Q e_\ell + P^{-1} \tau_\ell h_t$$

infinite vectors

$$e_1 = h_t P^{-1} \tau_0$$

$$e_2 = h_t (P^{-1} Q) P^{-1} \tau_0 + h_t P^{-1} \tau_1$$

$$e_3 = h_t (P^{-1} Q)^2 P^{-1} \tau_0 + h_t P^{-1} \tau_2 + h_t P^{-1} \tau_{\sim}$$

$\vdots$

$$e_\ell = h_t \sum (P^{-1} Q)^{\ell - m} P^{-1} \tau$$

$$e_{\ell+1} = (P^{-1}Q)\, e_\ell + h_t P^{-1} \tau_\ell$$

$$e_0 = 0$$

$$e_1 = (P^{-1}Q)\, e_0 + h_t P^{-1} \tau_0$$

$$= \underline{h_t P^{-1} \tau_0}$$

$$e_2 = (P^{-1}Q)\, e_1 + h_t P^{-1} \tau_1$$

$$= (P^{-1}Q)\, h_t P^{-1} \tau_0 + h_t P^{-1} \tau_1$$

$$e_3 = (P^{-1}Q)\, e_2 + h_t P^{-1} \tau_2$$

$$= (P^{-1}Q)\left((P^{-1}Q) h_t P^{-1} \tau_0 + h_t P^{-1} \tau_1\right) + h_t P^{-1} \tau_2$$

$$= h_t (P^{-1}Q)^2 P^{-1} \tau_0 + h_t (P^{-1}Q) P^{-1} \tau_1 + h_t P^{-1} \tau_2$$

$$e_\ell = h_t \sum_{m=1}^{\ell} (P^{-1}Q)^{\ell-m} P^{-1} \tau_{m-1}$$

want

$$\Rightarrow \|e_\ell\| \le \, ? \to 0$$

$$\Rightarrow \|e_\ell\| \le h_t \sum_{m=1}^{\ell} \|(P^{-1}Q)^{\ell-m} P^{-1} \tau_{m-1}\|$$

$$\le h_t \sum_{m=1}^{\ell} \|(P^{-1}Q)^{\ell-m} P^{-1}\| \, \|\tau_{m-1}\|$$

$$\|Ax\| \le \|A\| \, \|x\|$$

$$\|A\| \to \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

$$\|e_\ell\| \le h_t \sum_{m=1}^{\ell} \underbrace{\|(P^{-1}Q)^{\ell-m} P^{-1}\|}_{\text{stability} \; \le C} \underbrace{\|\tau_{m-1}\|}_{\text{consistency} \; \tau \to 0}$$

① A scheme is consistent if

$\tau_\ell \to 0$ as $h_t, h_x \to 0$

② A scheme is convergent if

$e_\ell \to 0$ as $h_t, h_x \to 0$

③ A scheme stable if

$$\underbrace{\|(P^{-1}Q)^{\ell-m} P^{-1}\|}\le C$$

want $\|(P^{-1}Q)^{\ell} P^{-1}\| \leq C$.

$\Rightarrow \leq \|(P^{-1}Q)\|^{\ell} \|P^{-1}\|$

$\Rightarrow$ need both $\|P^{-1}Q\| \leq 1$

$\|P^{-1}\| \leq C$

$$\frac{u_{k\,\ell+1} - u_{k\ell}}{h_t} + a\,\frac{u_{k\ell} - u_{k-1\,\ell}}{h_x} = 0$$

$$\gamma = \frac{a h_t}{h_x}$$

consistent. ✓

stable? ... sometimes

Lax: If consistent and stable Then convergent.

$$u_{k\,\ell+1} = \gamma\, u_{k-1\,\ell} + (1-\gamma)\, u_{k\ell}$$

$$P = I$$

$$Q = \operatorname{tridiag}(\gamma,\ 1-\gamma,\ 0)$$

$$= \begin{bmatrix} 1-\gamma & & & \\ \gamma & 1-\gamma & & 0 \\ & \gamma & \ddots & \\ & & \gamma & \ddots \end{bmatrix} \qquad \begin{matrix} \gamma \\ 1-\gamma \\ 0 \end{matrix} \begin{bmatrix} u_{k-1\,\ell} \\ u_{k\ell} \\ u_{k+1\,\ell} \end{bmatrix}$$

$$\|P^{-1}\| \overset{?}{=} \;= 1$$

$$P^{-1}Q = Q \Rightarrow \|Q\| \overset{?}{=}$$

$$\Rightarrow \quad \|Q\|_1 = \|Q\|_\infty = |\gamma| + |1-\gamma|$$

$$\Rightarrow \quad \text{if } 0 \leq \gamma \leq 1$$

$$\text{the } \|Q\|_1 \text{ or } \|Q\|_\infty = 1$$

$$\text{if } \gamma \notin [0,1]$$

$$\text{then } \|Q\| > 1$$

$$\Rightarrow \quad 0 \leq a\frac{h_t}{h_x} \leq 1 \Rightarrow h_t \leq \frac{h_x}{a}$$

Take infinite vector $\underline{x}$:

The Fourier Transform: $\hat{x}(\theta) = \sum_k x_k e^{-i\theta k}$

The inverse Fourier Transform:

$$x_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{x}(\theta)\, e^{i\theta k}\, d\theta$$

Parseval's Thm:

$$\boxed{\sum_k |x_k|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{x}(\theta)|^2\, d\theta}$$

So what?

$$\|P^{-1}Q\|^2 = \sup \frac{\|P^{-1}Qx\|^2}{\|x\|^2}$$

$$= \sup \frac{\|P^{-1}Qx\|^2}{\frac{h_x}{2\pi}\int_{-\pi}^{\pi}|\hat{x}(\theta)|^2\,d\theta}$$

P and Q are Toeplitz:

$\boxed{T \text{ is Toeplitz if } (T\underline{x})_j = \sum_k x_k t_{j-k} \text{ for some } \underline{t}}$

$\Rightarrow$ all diagonals are constant.

scipy $\to$ linalg $\to$ toeplitz.

scipy → linalg → toeplitz

$$T = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & & \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & & \\ & \ddots & & & & & & & & & & \\ & & & & & & -1 & 2 & -1 & & & \\ & & & & & & & & & \ddots & & \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

Find eigenvectors. Plot them.

[discretization of $-u_{xx}$]