Today 1/31

Objectives

1. Tie von Neumann stability to bounding $\| P^{-1} Q \|$

2. Make a new method: Lax-Wendroff

3. Quntify wave speed and wave diffusion

P & Q are toeplitz:

$$\boxed{\begin{array}{l}T \text{ is "toeplitz" if } (T\underline{x})_j = \sum_k x_k t_{j-k} \\ \text{for some toeplitz vector } t\end{array}}$$

→ diagonal are constant

→ $\underbrace{(Px)_j}_{y_j} = \sum_k x_k P_{j-k}$

[Derivation: pp 168–169]

Let $\hat{y}$ be the Fourier Transform of $\underline{y}$:

$$\hat{y}(\theta) = \sum_j y_j e^{-i\theta j}$$

$\underline{y}$ is given by $y_j = (Px)_j$

$$= \sum_k x_k p_{j-k}$$

$$\Rightarrow \quad \hat{y}(\theta) = \underset{\substack{\uparrow \\ \text{FT of } x}}{\hat{x}(\theta)} \; \underset{\substack{\uparrow \\ \text{FT of } p}}{\hat{p}(\theta)}$$

Also: $x = P^{-1}Px \Rightarrow \hat{x} = \frac{1}{\hat{p}} \cdot \hat{p} \cdot \hat{x}$

$\Rightarrow$ the FT of P^{-1} vector is $\dfrac{1}{\hat{p}}$

Back to $\|P^{-1}Q\|^2 = \sup \dfrac{\|P^{-1}Qx\|^2}{\|x\|^2}$

$$= \sup \frac{\|P^{-1}Qx\|^2}{\frac{h_x}{2\pi}\int_{-\pi}^{\pi} |\hat{x}(\theta)|^2 \, d\theta}$$ (Parseval)

$$= \sup \frac{\cancel{\frac{h_x}{2\pi}}\int_{-\pi}^{\pi} \left|\frac{\hat{g}}{\hat{p}}\hat{x}\right|^2 d\theta}{\cancel{\frac{h_x}{2\pi}}\int_{-\pi}^{\pi} |\hat{x}|^2 \, d\theta}$$

$$= \sup \frac{\max\left|\frac{\hat{g}}{\hat{p}}\right| \int_{-\pi}^{\pi} |\hat{x}|^2}{\int_{-\pi}^{\pi} |\hat{x}|^2}$$

$$= \max\left|\frac{\hat{g}}{\hat{p}}\right|$$

Take any Fourier mode:

$\underline{x}$ given by $x_k = \frac{1}{2\pi} e^{i\phi k}$

phi!!!

$$\Rightarrow \hat{x}(\theta) = \sum_k \boxed{x_k}\, e^{-i\theta k} \quad \text{(definition)}$$

$$= \frac{1}{2\pi} \sum_k e^{i\phi k} e^{-i\theta k}$$

$$= \frac{1}{2\pi} \sum_k e^{i(\phi-\theta)k}$$

(inverse) $\quad w_j = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{w}(\theta) e^{i\theta j} d\theta = \frac{1}{2\pi} e^{i\theta j}$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \delta(\phi-\theta)\, e^{i\theta j}\, d\theta$$

$$\Rightarrow \hat{w} = \delta(\phi-\theta)$$

the point?

$$x_k = e^{i\phi k} \qquad \text{Fourier mode.}$$

$$\Rightarrow \hat{x}(\theta) = \delta(\phi - \theta) \qquad \text{delta function}$$

$$\underset{\sim}{y} = P^{-1} Q \underset{\sim}{x}$$

$$\Rightarrow \hat{y} = \frac{\hat{q}(\theta)}{\hat{p}(\theta)} \delta(\phi - \theta)$$

$$\text{so} \quad y_k = \frac{\hat{q}(\theta)}{\hat{p}(\theta)} x_k$$

Any Fourier mode $x_n = \frac{1}{2\pi} e^{i\phi k}$

is an eigenvector of $P^{-1}Q$

and with eigenvalue $\frac{\hat{q}(\theta)}{\hat{p}(\theta)}$

Consider $P u_{k+1} = Q u_k + b_k h_t$

if $\max \left| \frac{\hat{q}(\theta)}{\hat{p}(\theta)} \right| \leq 1$

and $\max \left| \frac{1}{\hat{p}(\theta)} \right| \leq C$

then the scheme is

von Neumann stable

ETBS

$$\gamma = \frac{a h_t}{h_x}$$

$$u_{k,l+1} = \gamma\, u_{k-1,l} + (1-\gamma)\, u_{k,l}$$

$$P = I \qquad Q = \begin{bmatrix} 1-\gamma & & 0 \\ \gamma & \ddots & \\ 0 & \ddots & \ddots \end{bmatrix}$$

$$\Rightarrow \quad p_k = \begin{cases} 1 & k=0 \\ 0 & \text{else} \end{cases} \qquad q = \begin{cases} 1-\gamma & k=0 \\ \gamma & k=1 \\ 0 & \text{else} \end{cases}$$

recall $\hat{p}(\phi) = \sum_k p_k e^{-i\phi k}$

$$\Rightarrow \quad \hat{p} = 1$$

$$\Rightarrow \quad \hat{q} = \underbrace{\gamma e^{-i\phi}}_{k=1} + \underbrace{(1-\gamma)}_{k=0}$$

let $s = \dfrac{\hat{q}}{\hat{p}} = 1 - \gamma\left(1 - e^{-i\phi}\right)$

goal: bound $|s|$

$$\Rightarrow |s|^2 = \underbrace{\left(1-\gamma(1-e^{-i\phi})\right)}_{s} \cdot \underbrace{\left(1-\gamma(1-e^{i\phi})\right)}_{\bar{s}}$$

$$= 1-\gamma+\gamma e^{-i\phi} - \gamma + \gamma e^{i\phi}$$
$$+ \gamma^2 (1 - e^{-i\phi} - e^{i\phi} + 1)$$

$$= 1 - 2\gamma + \gamma \underbrace{(e^{-i\phi} + e^{i\phi})}_{2\cos\phi}$$
$$+ \gamma^2 (2 \underbrace{- e^{-i\phi} - e^{i\phi}}_{-2\cos\phi})$$

$$= 1 - 2\gamma + 2\gamma^2$$
$$+ 2(\gamma - \gamma^2)\cos\phi$$

$$\frac{\partial}{\partial\phi} = -2(\gamma-\gamma^2)\sin\phi \equiv 0$$

if $\phi = m\pi \quad m \in \mathbb{Z}$

Cases $\quad |s(\phi)|^2 = 1 - 2\gamma + 2\gamma^2 + 2(\gamma - \gamma^2)(\pm 1)$

$\uparrow \cos m\pi$

$+$ $\quad |s(\phi)|^2 = 1$

$-$ $\quad |s(\phi)|^2 = 1 - 2\gamma + 2\gamma^2 - 2\gamma + 2\gamma^2$

$$= 1 - 4\gamma + 4\gamma^2$$

$$= (1 - 2\gamma)^2$$

$\Rightarrow \quad |s(\phi)|^2 \leq 1$ iff $\quad |1 - 2\gamma| \leq 1$

$$\Rightarrow 0 \leq \gamma \leq 1$$

$$0 \leq \frac{a h_t}{h_x} \leq 1$$

$$u_{k\ell} = \lambda^{\ell} e^{i\theta k}$$

$\uparrow$ growth factor in time

$\downarrow$ plug into

$$u_{k,\ell+1} = \gamma u_{k-1,\ell} + (1-\gamma) u_{k,\ell}$$

Expand $u(x,t)$ in time:

$$\boxed{u(x,t+h_t) = u(x,t) + h_t\, u_t(x,t)} + \frac{h_t^2}{2} u_{tt}(x,t) + \cancel{O(h_t^3)}$$

$O(h_t^2)$

$$\Rightarrow \frac{u(x,t+h_t) - u(x,t)}{h_t} = u_t + \frac{h_t}{2} u_{tt}$$

$u_t + a u_x = 0 \Rightarrow u_{xt} = -a u_{xx}$

$\Rightarrow u_t = -a u_x$

$$\Rightarrow \frac{u(x,t+h_t) - u(x,t)}{h_t} + a u_x - \frac{h_t}{2} u_{tt} = 0$$

$u_{tt} = -a u_{xt}$

$= a^2 u_{xx}$

Fwd in time   center space   2nd order central

$$\frac{u_{k,\ell+1} - u_{k,\ell}}{h_t} + a \frac{u_{k+1,\ell} - u_{k-1,\ell}}{2h_x} - \frac{a^2 h_t}{2} \frac{u_{k-1,\ell} - 2u_{k,\ell} + u_{k+1,\ell}}{h_x^2}$$

Lax-Wendroff.

Why does the solution "smooth" out?

$$z(x,t) = z_0 \, e^{i(kx - \omega t)}$$

z_0 amplitude

k wave #

ω ang frequency

T = period

$= \frac{1}{\nu} \leftarrow$ frequency

$\omega = 2\pi\nu$

λ = wave length

$= \frac{2\pi}{k}$

$$z = z_0 e^{i(kx - \omega t)}$$

Let L be <u>any</u> linear operator:

$$Lz = \lambda z$$

example: $L = \partial_t$

$$Lz = \partial_t z = -i\omega z_0 e^{i(kx-\omega t)}$$

$$= -i\omega z$$

<u>example</u>

$$L = \partial_x$$

$$Lz = \partial_x z = ikz$$

$$\boxed{z \text{ solve } Lz = 0 \text{ iff } \lambda = 0}$$

This is the "dispersion" relation

Example (advection)

$$L = \partial_t + a\,\partial_x$$

$$\to \quad Lz = -i\omega z + aikz$$

$$= (-i\omega + aik)\,z$$

$$= i(ak-\omega)\,z$$

The dispersion relation is when $\lambda = 0$.

$$\Rightarrow \quad i(ak-\omega) = 0$$

$$\Rightarrow \quad \omega = ak$$

Example

$$L = \partial_t + a\,\partial_x - D\,\partial_{xx}$$

$$\Rightarrow \quad Lz = -i\omega z + aik z - D(ik)^2 z$$

$$= (-i\omega + aik + Dk^2)\, z$$

$$\lambda \equiv 0$$

$$\underset{(-i)}{\Rightarrow} \quad i\omega = aik + Dk^2$$

$$\Rightarrow \quad \omega = ak - Dik^2$$

$$\omega = \omega(k)$$

$$z = z_0 e^{i(kx - \omega t)}$$

let $\omega = \omega(k)$

$$= \underset{\text{real}}{\alpha(k)} + i\,\underset{\text{imag}}{\beta(k)}$$

$$= z_0 e^{i(kx - \alpha t - i\beta t)}$$

$$= z_0 \underbrace{e^{\beta t}}_{\text{dissipitive}} e^{i(kx - \alpha t)}$$