Today 2/5

1. Quantify dispersion and dissipation

2. Introduce "conservation laws"

Last time

let $z(x,t) = z_0 e^{i(kx - \omega t)}$

solve $Lu = 0$ (e.g. $L = \partial_t + a\partial_x$)

with $\omega(k) = \alpha(k) + i\beta(k)$

if $\beta < 0 \longrightarrow$ dissipation

if $\alpha(k)$ = nonlinear in k

$\longrightarrow$ dispersive

(if $\alpha(k) = c \cdot k$
then non dispersive)

($v_{ph} = \frac{\alpha(k)}{k}$ = phase velocity)

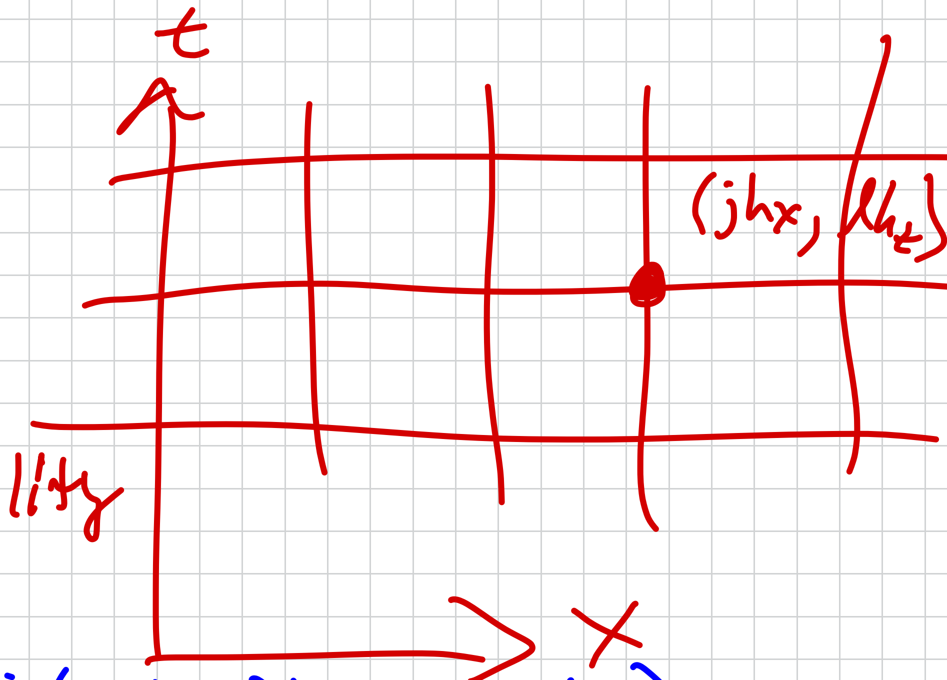
So what about our method?

let $z_{j\ell} = z_0 e^{i(kjh_x - \omega \ell h_t)}$

look at ETBS

$$u_{j\ell+1} = \gamma u_{j-1\ell} + (1-\gamma) u_{j\ell}$$

need $\gamma = \frac{a h_t}{h_x} < 1$ for stability



$$\Rightarrow z_0 e^{i(kjh_x - \omega(\ell+1)h_t)} = z_0 \gamma e^{i(k(j-1)h_x - \omega \ell h_t)} + (1-\gamma) z_0 e^{i(kjh_x - \omega \ell h_t)}$$

$$\Rightarrow e^{-\omega h_t} = \gamma e^{-ikh_x} + (1-\gamma)$$

$$e^{-i\omega h_t} = \underbrace{\gamma e^{-ikh_x} + 1 - \gamma}_{\text{the "symbol" } s(kh_x) \text{ or the amplification factor}}$$

$$= s(kh_x)$$

What is "ω" for <u>this</u> numerical scheme?

$$e^{-i\omega h_t} = s$$

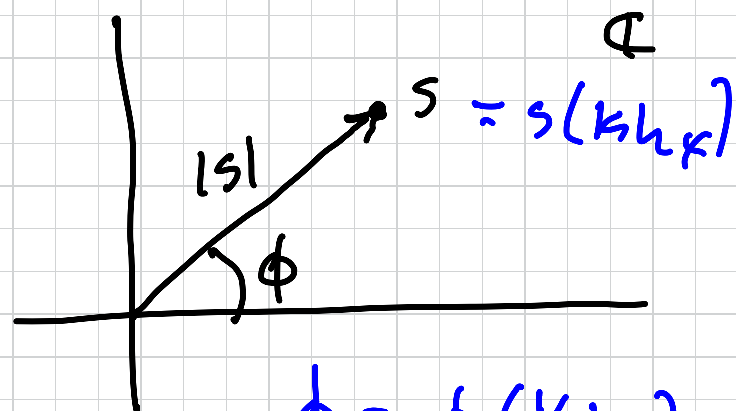
let s = complex

$$= |s| e^{i\phi} \quad \text{for some } \phi$$

$$= e^{\ln|s| + i\phi}$$

$$\Rightarrow \quad -i\omega h_t = \ln|s| + i\phi$$

$$\Rightarrow \quad \omega = \frac{i\ln|s| - \phi}{h_t}$$



$$\phi = \phi(kh_x)$$

$$\omega = \frac{i\ln|s| - \phi}{h_t}$$

$$z_{j\ell} = z_0 e^{i(k_j h_x - \omega \ell h_t)}$$

$$= z_0 e^{i\left(k_j h_x - \frac{i\ln|s| - \phi}{h_t}\ell h_t\right)}$$

$$= z_0 e^{\ln|s|\ell} e^{i\left(k_j h_x - \left(\frac{-\phi}{h_t}\right)\ell h_t\right)}$$

$$= z_0 |s|^{\ell} e^{i\left(k_j h_x - \underbrace{\left(\frac{-\phi}{h_t}\right)}\ell h_t\right)}$$

if $|s| < 1$ then dissipitive

if $-\frac{\phi}{h_t}$ nonlinear in k,

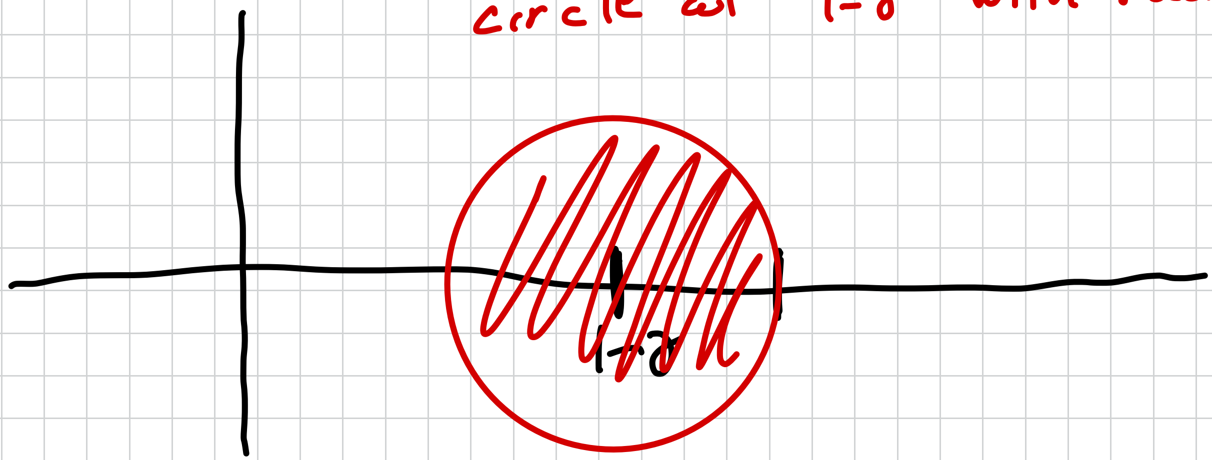
then dispersive.

ETBS

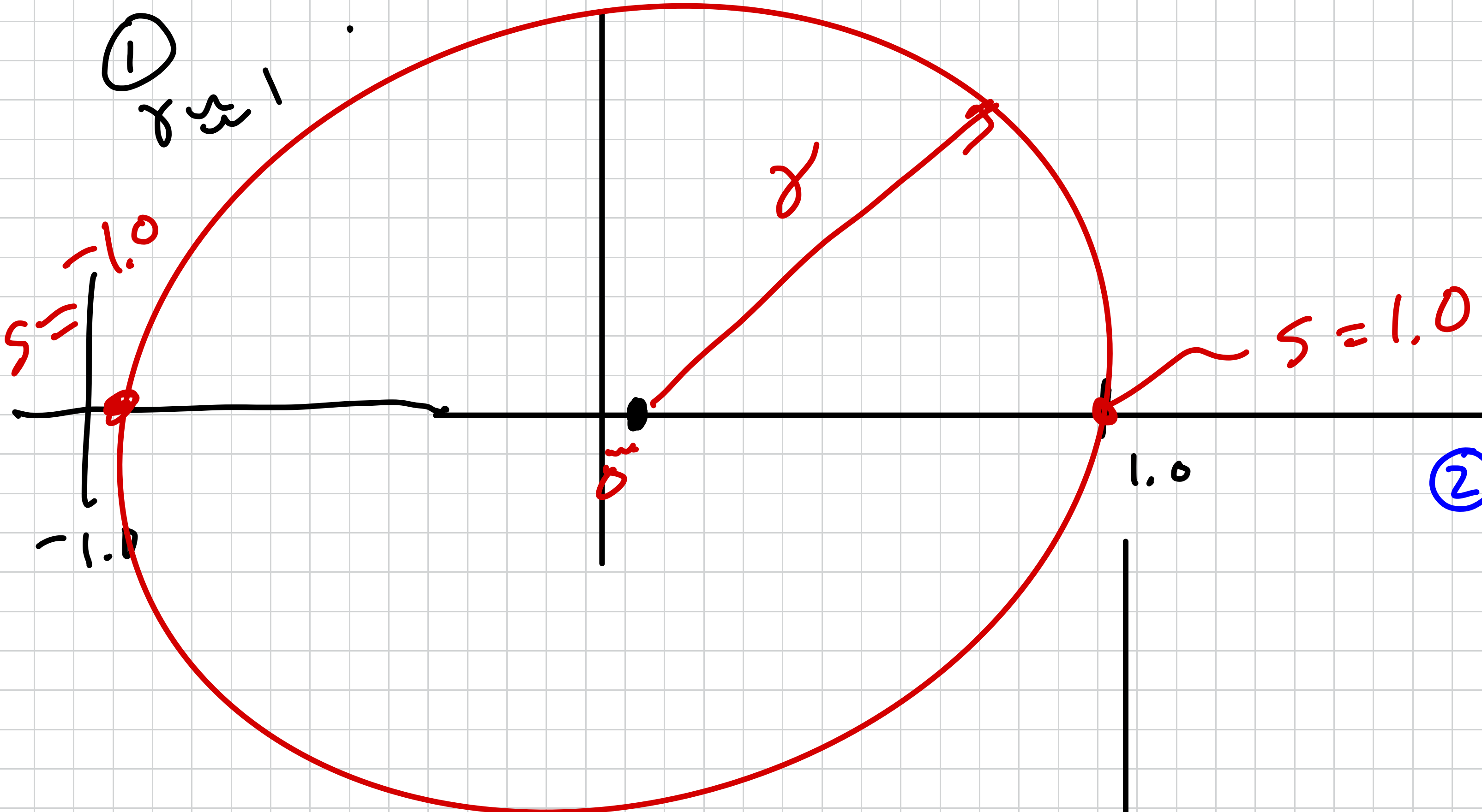
$$e^{-i\omega h_t} = s(k h_x)$$

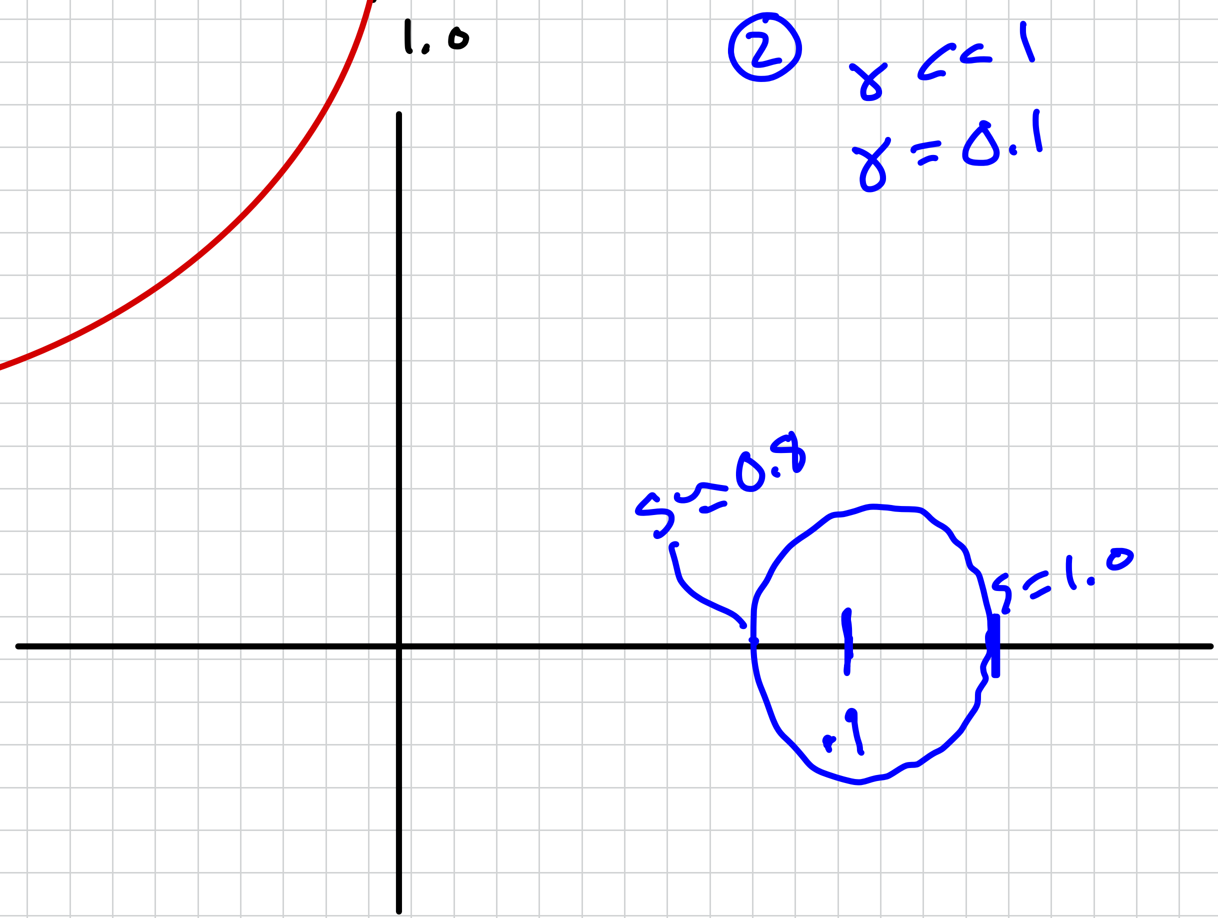
$$= \underbrace{1-\gamma + \gamma e^{-ikh_x}}_{\text{circle at } 1-\gamma \text{ with radius } \gamma}$$

$$\gamma = a\frac{h_t}{h_x}$$



$$s = 1 - \gamma + \gamma e^{-ikh_x}$$





$$s(kh_x) = 1 - \gamma + \gamma e^{-ikh_x}$$

if $kh_x \simeq$ small

then $e^{-ikh_x} \simeq 1 - ikh_x$

$$\Rightarrow s \simeq 1 - \gamma + \gamma - \gamma i k h_x$$
$$= 1 - \gamma i k h_x$$

Return to $e^{-i\omega h_t} = s$

$$\Rightarrow \cos \omega h_t - i \sin \omega h_t \simeq 1 - \gamma i k h_x$$

if $\omega h_t \simeq$ small

$$1 - i\omega h_t \simeq 1 - \gamma i k h_x$$

$$\Rightarrow \omega \simeq \gamma k \frac{h_x}{h_t} = ak$$

$$u_t + a u_x = 0$$

what about non-constant a?

what if this is nonlinear?

what if we have more than a single scalar variable? (Euler)

what about 2D? 3D?

A conservation law is of the form

$$\frac{\partial}{\partial t} u + \frac{\partial f(u)}{\partial x} = 0$$

If f is differentiable

Then $u_t + f'(u)\, u_x = 0$

In 2D or 3D:

$$\frac{\partial u}{\partial t} + \nabla_{\underline{x}} \cdot \left(\underline{f}(u)\right) = 0$$

Example

$$u_t + y u_x - x u_y = 0$$

Conservative form?

$$u_t + \nabla \cdot (\quad) = 0 \quad ?$$

hint: $\nabla \cdot (\underset{\sim}{b} u) = \underline{b} \cdot \nabla u + (\nabla \cdot \underset{\sim}{b}) u$

$$u_t + [y, -x]^T \cdot \nabla u \qquad \nabla \cdot \begin{bmatrix} y \\ -x \end{bmatrix} = 0$$

$$\Rightarrow u_t + \nabla \cdot \left(\begin{bmatrix} y \\ -x \end{bmatrix} u \right) = 0$$

1D conservative form

$$u_t + \partial_x (f(u)) = 0$$

advection:

$$u_t + \partial_x (a u) = 0$$

non conservative form:

$$u_t + f'(u) \partial_x u = 0$$

advection

$$u_t + a \partial_x u = 0$$

Example

$$u_t + x u_x = 0$$

2D

$$u_t + \nabla \cdot (\underline{f}(u)) = 0$$

$$u_t + [\partial_x, \partial_y] \cdot [f_1(u), f_2(u)] = 0$$

1D

$$u_t + \partial_x(f(u)) = 0$$

Case $\quad f(u) = \frac{u^2}{2}$ Burger's eq.

$$u_t + \partial_x\left(\frac{u^2}{2}\right) = 0$$

or

$$u_t + u u_x = 0$$

Consider any curve $x(t)$ that satisfies $x'(t) = u(x(t), t)$

$$x(0) = u(x(0), 0)$$

$$\frac{d\, u(x(t), t)}{dt} = \frac{\partial (u(x(t), t))}{\partial t} + \frac{dx(t)}{dt} \cdot \frac{\partial u(x(t), t)}{\partial x}$$

$$= u_t + x'(t)\, u_x = 0$$

Curves given by

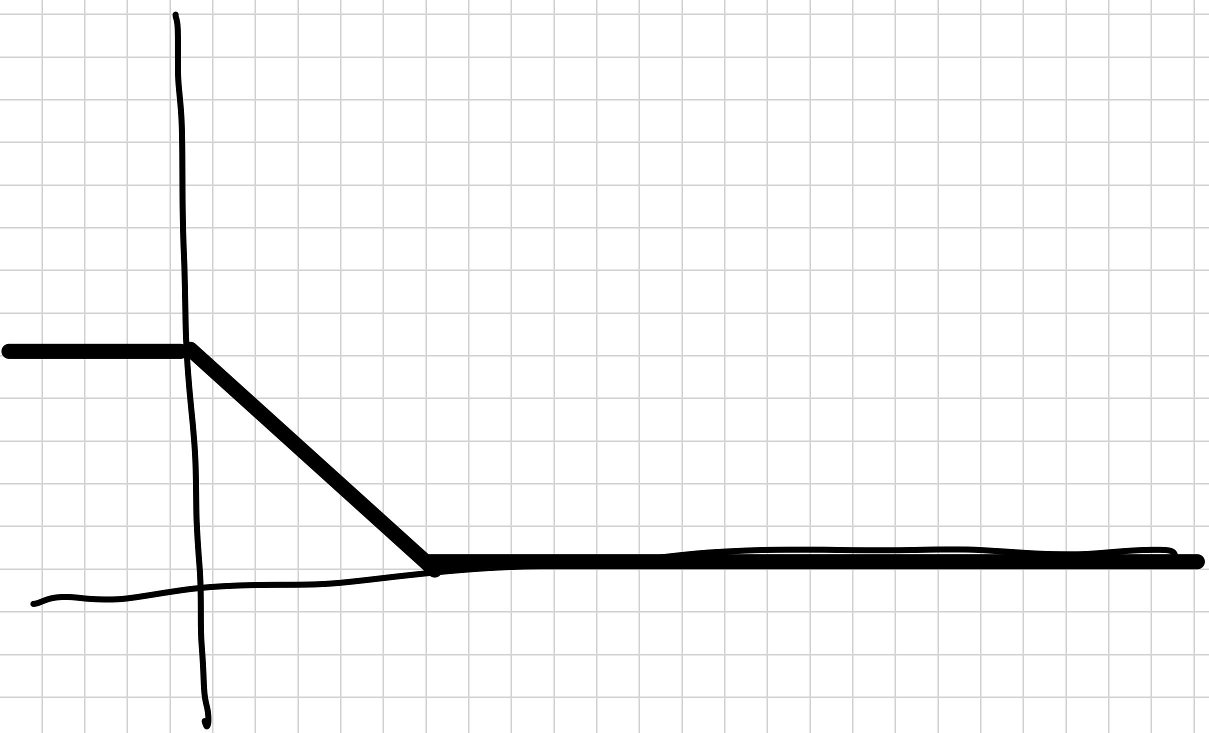
$$\frac{dx}{dt} = u \leftarrow \text{an initial cond}$$

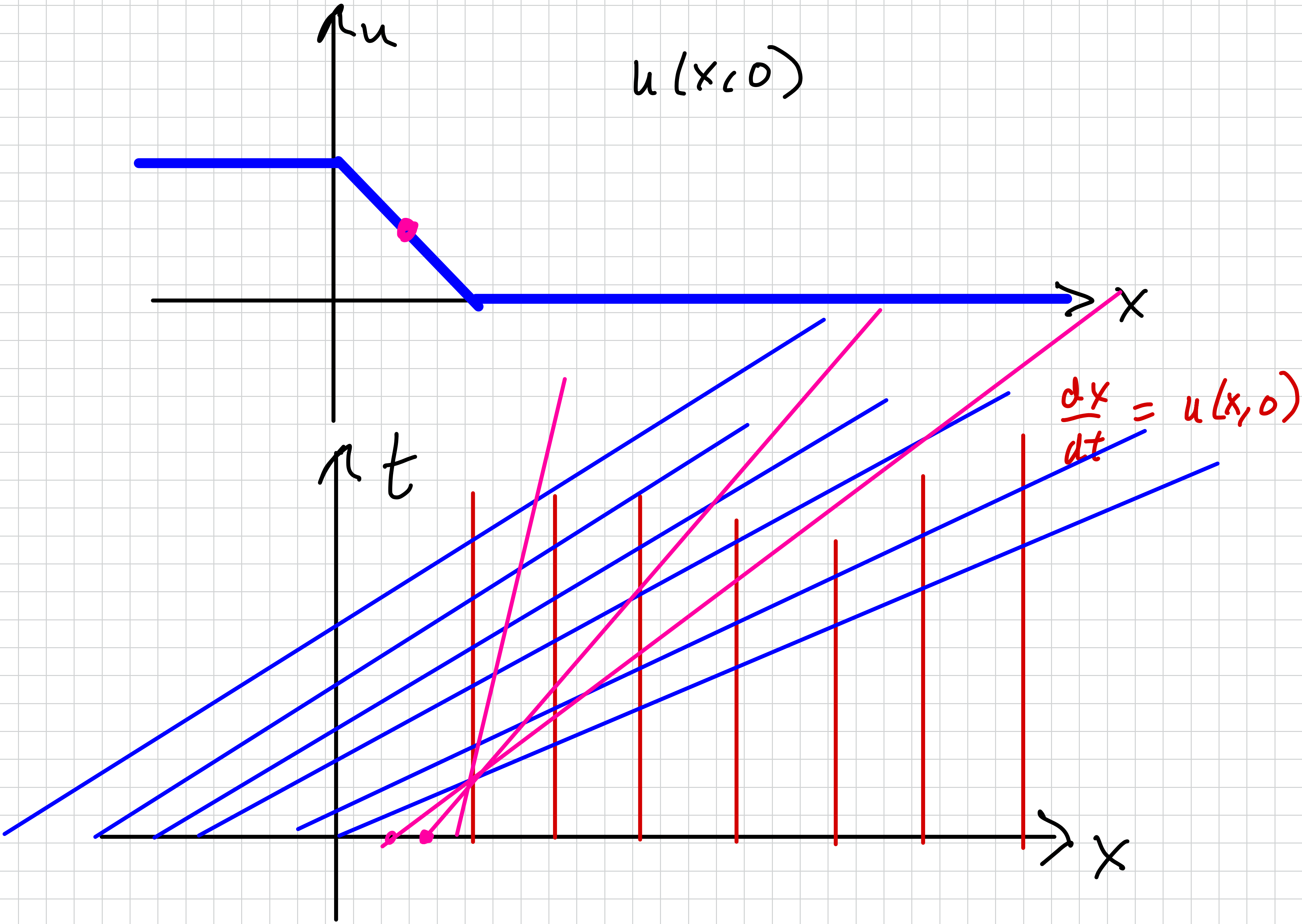
$\Rightarrow$ straight lines that depend on $u(x,0)$

$\boxed{\text{Case 1}}$

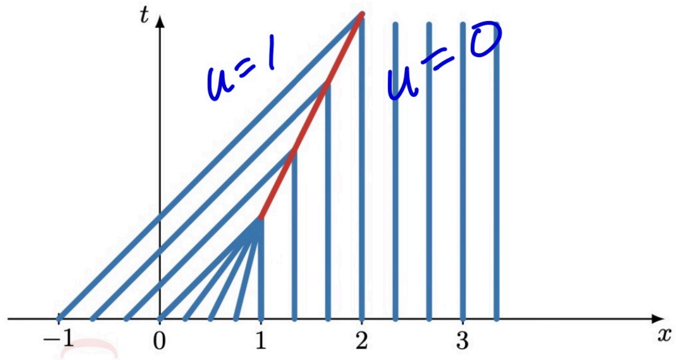
$$u_t + uu_x = 0$$

$$u(x,0) = \begin{cases} 1 & x \leq 0 \\ 1-x & x \in [0,1] \\ 0 & x > 1 \end{cases}$$





u
u(x,0)
x
t
dx/dt = u(x,0)
x



t
x
−1
0
1
2
3
u=1
u=0