Today 2/7

1) Look at characteristic curves

2) Derive a conservation

3) Introduce a Finite Volume "framework"

Take $\dfrac{\partial u}{\partial t} + \dfrac{\partial (f(u))}{\partial x} = 0$

$\Rightarrow \quad \dfrac{\partial u}{\partial t} + f'(u)\dfrac{\partial u}{\partial x} = 0$

Let $x(t)$ be a curve such that

$$\frac{dx(t)}{dt} = f'(u)$$

Then $\dfrac{du(x(t),t)}{dt} = \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x}\dfrac{dx}{dt}$

$= u_t + f'(u)\, u_x$

$= 0$

$$\Rightarrow \begin{cases} \dfrac{du}{dt} = 0 \\ \dfrac{dx}{dt} = 0 \end{cases} \Rightarrow \quad \begin{aligned} u &= c \\ x &= ut + d \end{aligned}$$

$\Rightarrow$ straight lines

each curve is

$$x(t) = u(x_0, 0) \cdot t + x_0$$

from each x_0

Cases

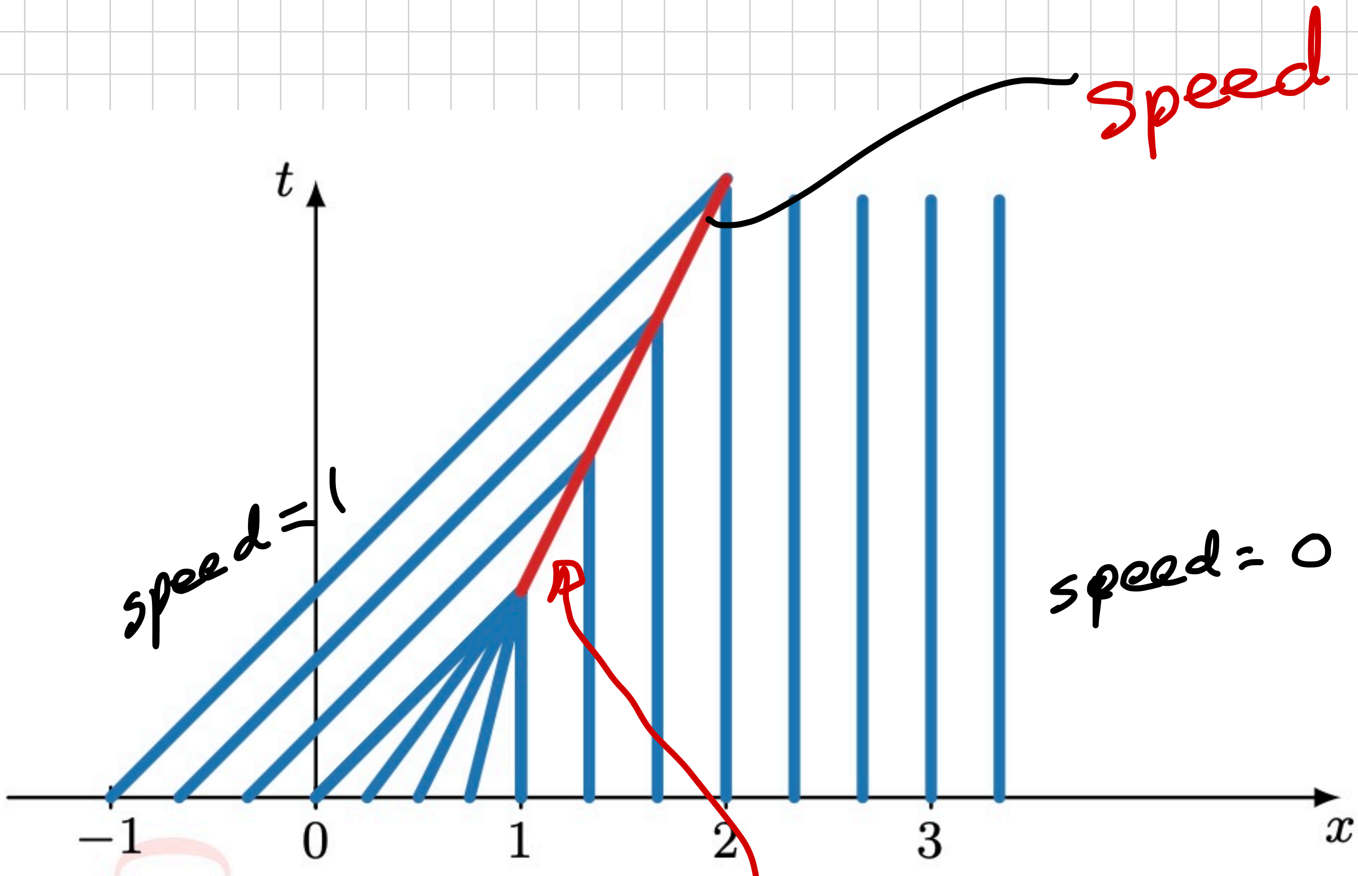
① Shock

$$u_0(x,0) = \begin{cases} 1 & x < 0 \\ 1-x & x \in [0,1] \\ 0 & x > 1 \end{cases}$$

② Rarefaction

$$u_0(x,0) = \begin{cases} 0 & x < 0 \\ x & x \in [0,1] \\ 1 & x > 1 \end{cases}$$

$\boxed{\text{Draw the characteristic curves}}$



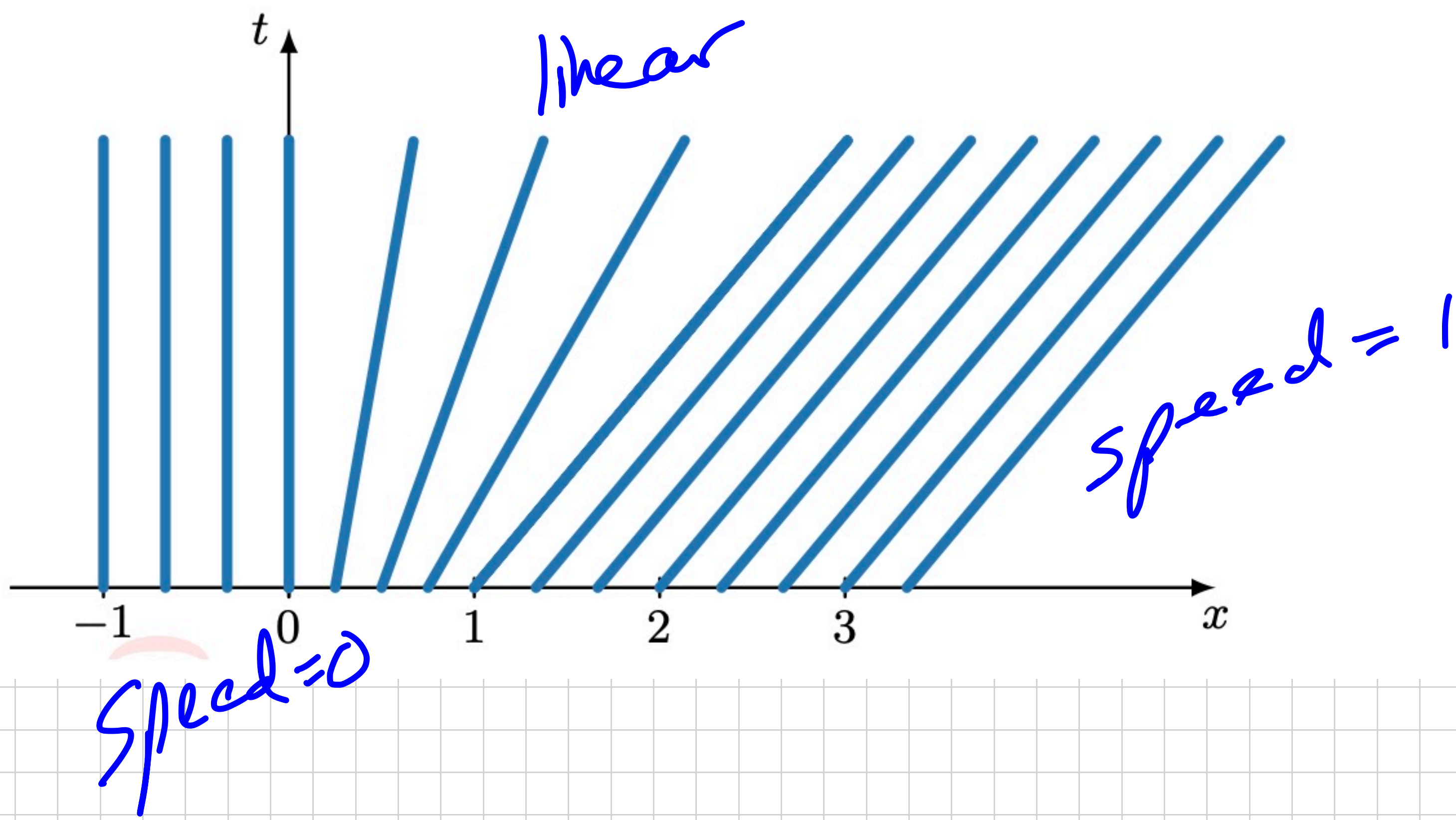
let $\hat{x}(t)$ = curve

then $\hat{x}'(t) = \dfrac{f(u^+) - f(u^-)}{u^+ - u^-}$

on the right (for u^+), on the left (for u^-)

Burgers: $f(u) = \frac{u^2}{2}$

$$= \frac{\frac{0^2}{2} - \frac{1^2}{2}}{0 - 1} = 1/2$$


t
linear
speed = 1
−1
0
1
2
3
x
Speed=0

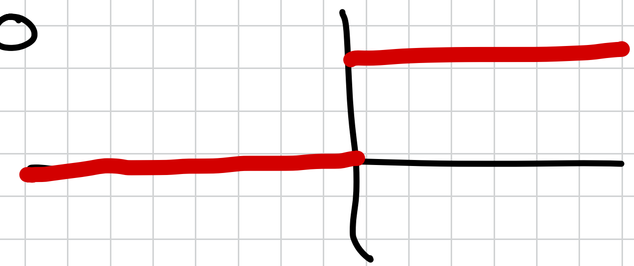
Definition 6.5: Riemann Problem

Consider scalar conservation law (6.13) with initial condition

$$u(x,0) = \begin{cases} u^- & \text{if } x \leq 0, \\ u^+ & \text{if } x > 0. \end{cases} \tag{6.29}$$

This problem consisting of constant left and right states separated by a jump discontinuity is called a Riemann problem.

$$u(x,0) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$



a. Unphysical weak solution with entropy-violating shock.

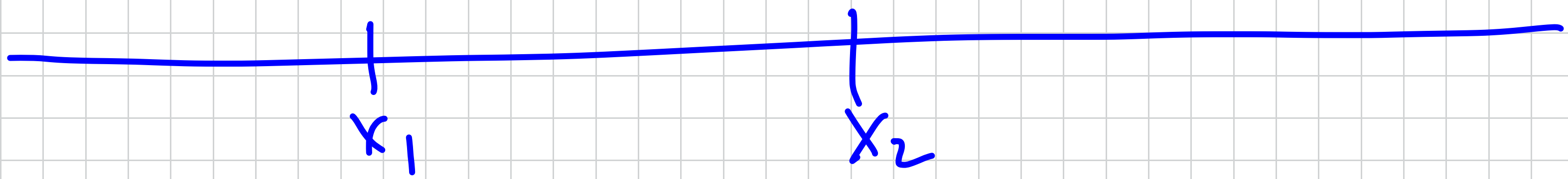
b. Unique physically relevant, vanishing viscosity weak solution (rarefaction wave).

Return to conservation law:

Conserved quantity.

$u(x,t)$ = density at x at time t. $\left[\frac{kg}{m}\right]$

$\int_{x}^{x_2} u(x,t)\,dx$ = total mass in $[x_1,x_2]$ at time t.



mass can only change due to flow in or out at x_1 or x_2

"flux"

$F_1(t)$ = rate at x_1 $\left[\frac{kg}{s}\right]$

$F_2(t)$ = rate at x_2

(flux "to the right" $F_i(t) > 0$)

$$\frac{d}{dt}\int_{x_1}^{x_2} u(x,t)\,dx = \underbrace{F_1(t)}_{\text{flow in at } x_1} - \underbrace{F_2(t)}_{\text{flow in at } x_2}$$

general

$$= f(u(x_1,t)) - f(u(x_2,t))$$

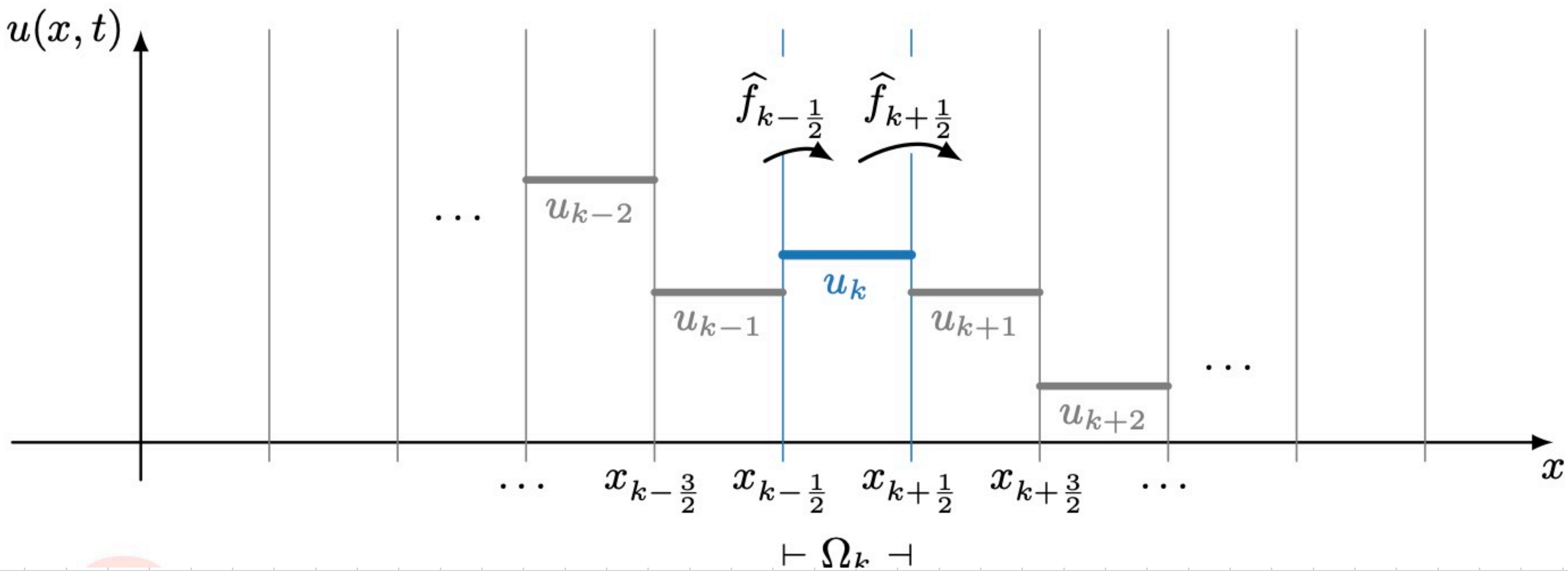
$$= -f(u(x,t))\Big|_{x_1}^{x_2}$$

(Assume smooth)

$$\Rightarrow \frac{d}{dt}\int_{x_1}^{x_2} u(x,t)\,dx + \int_{x_1}^{x_2} \frac{d}{dx} f(u(x,t))\,dx = 0$$

$$\Rightarrow \int_{x_1}^{x_2} u_t + (f(u))_x \, dx = 0$$

Idea



$$\Omega_k = [x_{k-1/2}, x_{k+1/2}]$$

$$\int_{\Omega_k} \frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} \, dx = 0$$

$$h_x \frac{d \bar{u}_k(t)}{dt} + f(u(x_{k+1/2}, t)) - f(u(x_{k-1/2}, t)) = 0$$

$$\bar{u}_k(t) = \frac{1}{h_x} \int_{\Omega_k} u(x,t) \, dx$$

the solution $u(x,t)$ in Ω_k satisfies

$$\frac{d\,\bar{u}_k(t)}{dt} + \frac{f(u(x_{k+1/2},t)) - f(u(x_{k-1/2},t))}{h_x} = 0$$

exactly.

To turn into a method:

1. discrete values $u_k \simeq \bar{u}_k(t)$
2. define a numerical flux:
$$\hat{f}_{k+1/2} \simeq f(u(x_{k+1/2},t))$$
3. ODE solver

Method "framework":

$$\frac{d\, u_k(t)}{dt} + \frac{\hat{f}_{k+1/2}(t) - \hat{f}_{k-1/2}(t)}{h_x} = 0$$

ODE solver

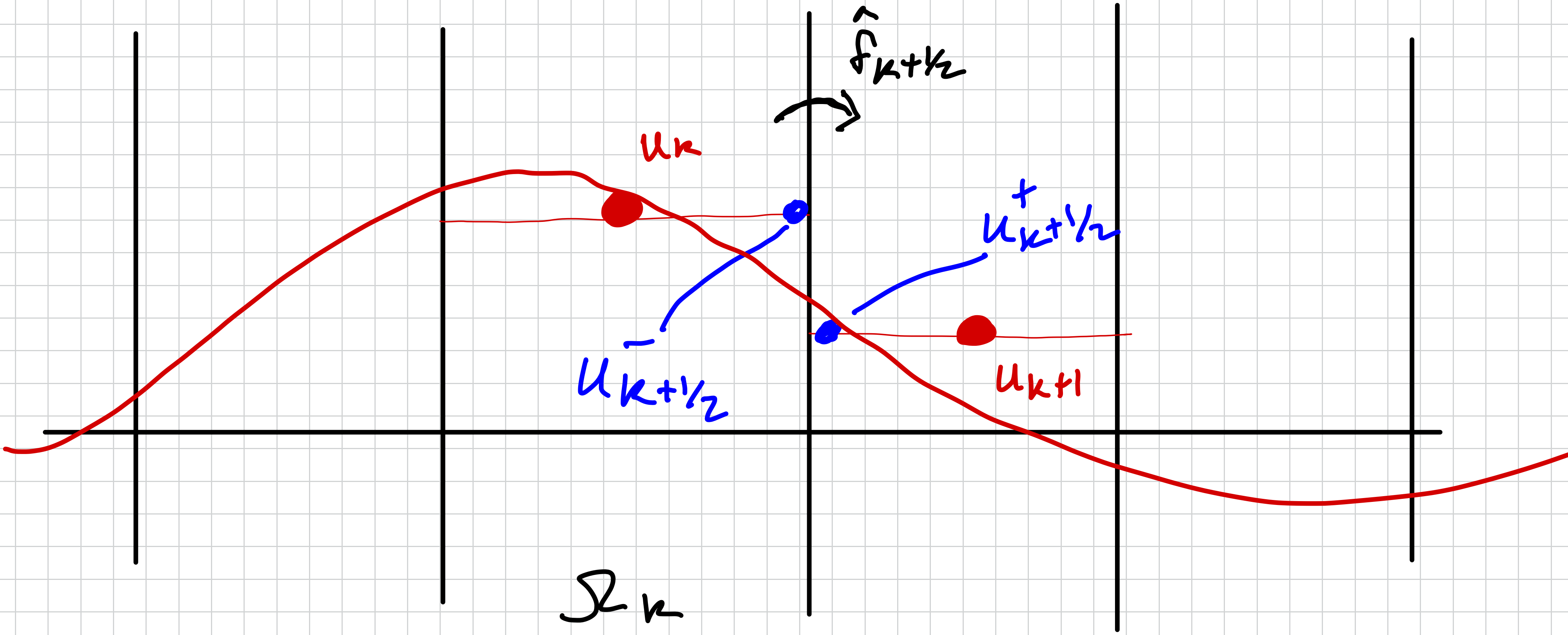
FWD Euler
RK4
whatever

For $\hat{f}_{k+1/2}(t)$.

We will use $\hat{f}_{k+1/2}(t) \approx f^*(\cdot, \cdot)$

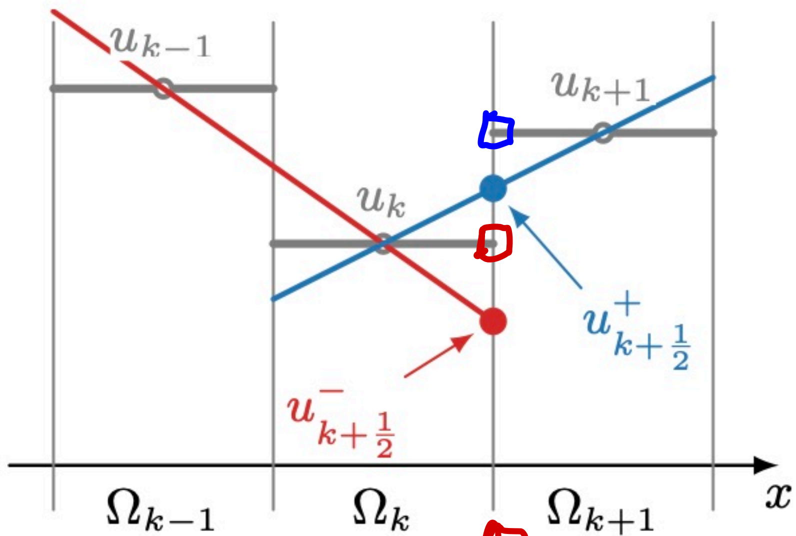
$$= f^*\left(u^-_{k+1/2}(t),\ u^+_{k+1/2}(t)\right)$$

approx value "just to the left" of $x_{k+1/2}$

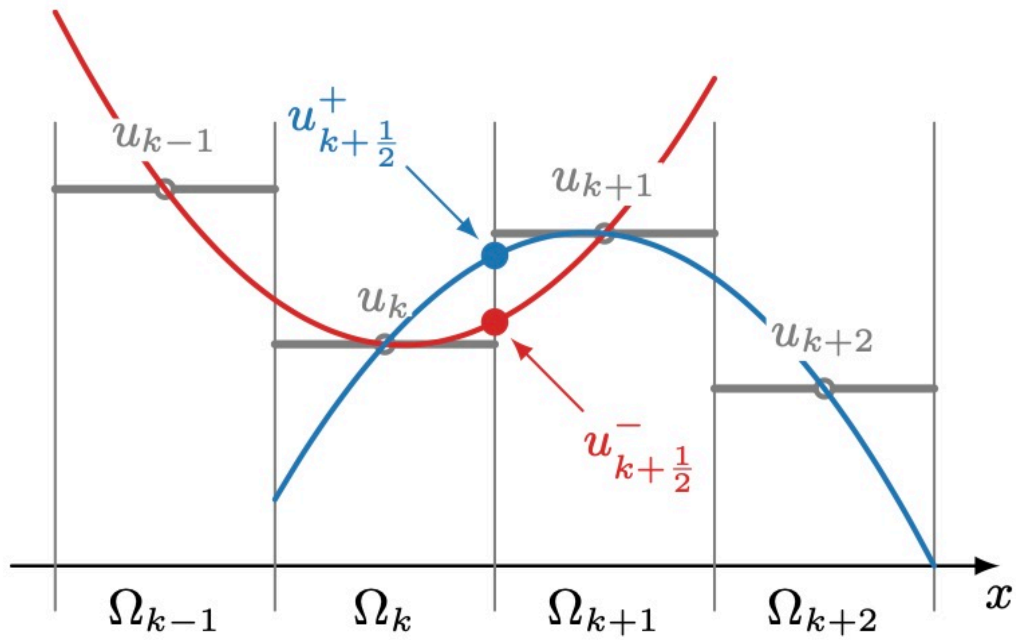


easiest: $u^-_{k+1/2} = u_k$

$$u^+_{k+1/2} = u_{k+1}$$



u_{k-1}
u_k
u_{k+1}
$u^+_{k+\frac{1}{2}}$
$u^-_{k+\frac{1}{2}}$
Ω_{k-1}
Ω_k
Ω_{k+1}
x



u_{k-1}
$u^+_{k+\frac{1}{2}}$
u_{k+1}
u_k
u_{k+2}
$u^-_{k+\frac{1}{2}}$
Ω_{k-1}
Ω_k
Ω_{k+1}
Ω_{k+2}
x

$X_{k+1/2}$

let $u_{k,\ell}$ = approx cell average at t

+

FWD Euler

$$\frac{u_{k\ell+1} - u_{k\ell}}{h_t} + \frac{f^*(u_{k\ell}, u_{k+1\ell}) - f^*(u_{k-1\ell}, u_{k\ell})}{h_x} = 0$$

The PDE gives us $f(u)$

Definition 6.8: Consistent numerical flux function

A numerical flux function $f^*(u^-, u^+)$ for conservation law (6.40) is called consistent if

- $f^*(u, u) = f(u) \qquad \forall\, u$
- $f^*(u^-, u^+)$ *is Lipschitz continuous in each argument.*

advection

$$u_t + (au)_x = 0$$

$$f(u) = a \cdot u$$

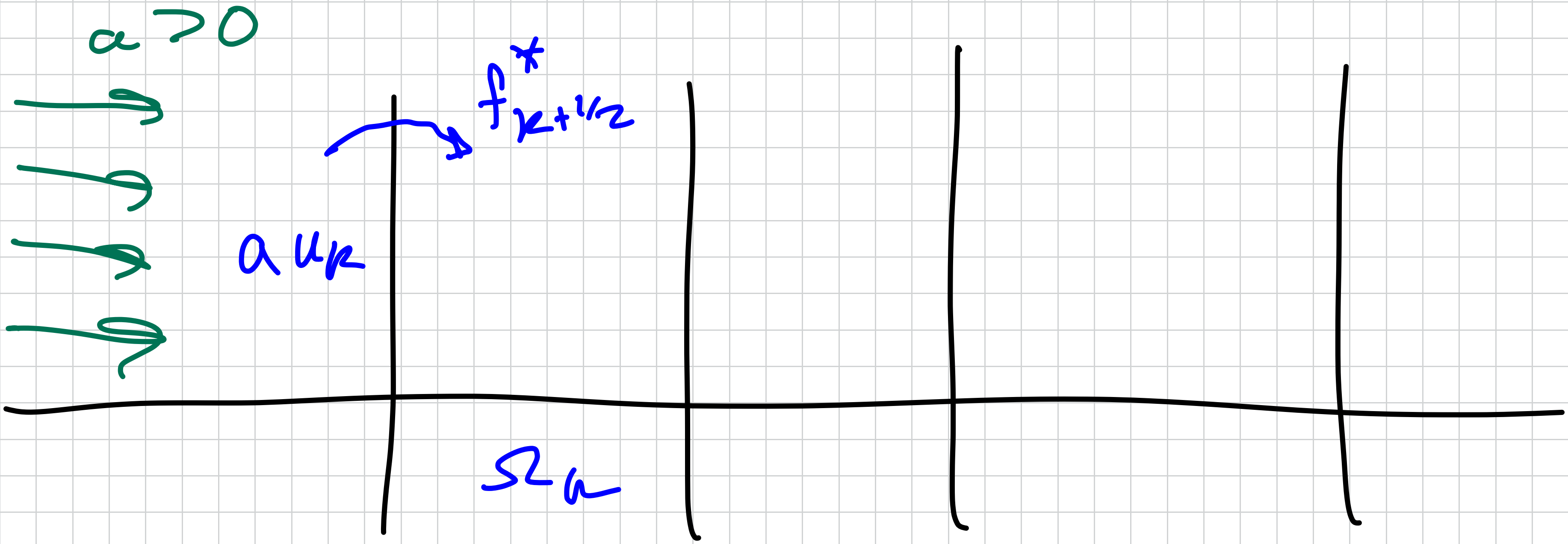
Pick $f^*(u_{k\ell}, u_{k+1\ell}) = a\, u_{k\ell}$

and $f^*(u_{k-1\ell}, u_{k\ell}) = a\, u_{k-1\ell}$

$$0 = \frac{u_{k\ell+1} - u_{k\ell}}{h_t} + \frac{f^*(u_{k\ell}, u_{k+1\ell}) - f^*(u_{k-1\ell}, u_{k\ell})}{h_x}$$

$$\Rightarrow \frac{u_{k\ell+1} - u_{k\ell}}{h_t} + a\,\frac{u_{k\ell} - u_{k-1\ell}}{h_t} = 0$$

assume $a > 0$



$\alpha > 0$
$a u_k$
$f^*_{k+1/2}$
Ω_k

if $a > 0$: $f^*(u_{k\ell}, u_{k+1\ell}) = a u_{k\ell}$

if $a < 0$: $f^*(u_{k\ell}, u_{k+1\ell}) = a u_{k+1\ell}$

Q: what if we want f^* to "work" for both?

hint: use $|a|$

$$\rightarrow f^*(u_{k\ell}, u_{k+1\ell}) = a\left(\frac{u_{k\ell} + u_{k+1\ell}}{2}\right) - \frac{|a|}{2}\left(u_{k+1\ell} - u_{k\ell}\right)$$