Today 2/7

1) Look at characteristic curves

2) Derive a conservation

3) Introduce a Finite Volume "framework"
Take \[ \frac{\delta u}{\delta t} + \frac{\partial (f(u))}{\partial x} = 0 \]

\[ A \]

Let \( x(t) \) be a curve such that \[ \frac{dx(t)}{dt} = f'(u) \]

Then \[ \frac{du(x(t),t)}{dt} = \frac{\delta u}{\delta t} + \frac{\delta u}{\delta x} \frac{dx}{dt} \]

\[ = u_t + f'(u) u_x \]

\[ = 0 \]

\[ \frac{du}{dt} = 0 \]
\[ \frac{dx}{dt} = 0 \]

\[ u = c \]
\[ x = ut + d \]
\[ \Rightarrow \text{straight line} \]
each curve is

\[ x(t) = u(x_0, 0) \cdot t + x_0 \]

from each \( x_0 \)

\[ u_0(x, 0) = \begin{cases} 
1 & x < 0 \\
1 - x & x \in [0, 1] \\
0 & x > 1 
\end{cases} \]

**Cases**

1. **Shock**

2. **Rarefaction**

Draw the characteristics for curves.
Burgers: \( f(u) = \frac{u^2}{2} \)

Let \( \mathcal{X}(\xi) = \text{curve} \)

Then \( \mathcal{X}'(\xi) = \frac{f(u^+) - f(u^-)}{u^+ - u^-} \)

\( = \frac{0^2 - 1^2}{0-1} = \frac{1}{2} \)
**Definition 6.5: Riemann Problem**

Consider scalar conservation law (6.13) with initial condition

\[
 u(x, 0) = \begin{cases} 
 u^- & \text{if } x \leq 0, \\
 u^+ & \text{if } x > 0.
\end{cases} \quad (6.29)
\]

This problem consisting of constant left and right states separated by a jump discontinuity is called a Riemann problem.

\[
 u(x, t) = \begin{cases} 
 0 & x \leq 0 \\
 1 & x > 0
\end{cases}
\]

**a.** Unphysical weak solution with entropy-violating shock.

**b.** Unique physically relevant, vanishing viscosity weak solution (rarefaction wave).
Return to conservation law:

Conserved quantity.

\[ u(x,t) = \text{density at } x \text{ at time } t. \frac{[kg]}{m} \]

\[ \int_{x_1}^{x_2} u(x,t) \, dx = \text{total mass } m \text{ at } x, \text{ at time } t. \]

Mass can only change due to flow in or out at \( x_1 \) or \( x_2 \).

“Flux”

\[ F_1(t) = \text{rate at } x_1 \frac{[kg]}{s} \]

\[ F_2(t) = \text{rate at } x_2 \frac{[kg]}{s} \]

(flux “to the right” \( F_1(t) > 0 \))
\[
\frac{d}{dt} \int_{x_1}^{x_2} u(x,t) \, dx = \frac{F_1(t)}{\text{flow in at } x_1} - \frac{F_2(t)}{\text{flow out at } x_2}
\]

\[= f(u(x_1,t)) - f(u(x_2,t)) \]

\[= -\int_{x_1}^{x_2} \frac{d}{dx} \left( f(u(x,t)) \right) \, dx \]

\[= 0
\]

(Assume smooth)
\[ \hat{f}_{k-\frac{1}{2}} \quad \hat{f}_{k+\frac{1}{2}} \]

\[ u(x, t) \]

\[ \cdots \quad x_k-\frac{3}{2} \quad x_k-\frac{1}{2} \quad x_k+\frac{1}{2} \quad x_k+\frac{3}{2} \quad \cdots \]

\[ u_{k-2} \quad u_k \quad u_{k+1} \quad u_{k+2} \]

\[ \cdots \quad \Omega_k \quad \cdots \]

\[ \mathbb{R}_k = [x_{k-\nu}, x_{k+\nu}] \]

\[ \int_{\mathbb{R}_k} \frac{du}{dt} + \frac{\partial f(u)}{\partial x} \, dx = 0 \]

\[ h_x \cdot \frac{d\overline{u}_k(t)}{dt} + f(u(x_{k+\nu}, t)) - f(u(x_{k-\nu}, t)) = 0 \]

\[ \overline{u}_k(t) = \frac{1}{h_x} \int_{\mathbb{R}_k} u(x, t) \, dx \]
The solution $u(x,t)$ in $\Omega_k$ satisfies

$$\frac{d}{dt} \bar{u}_k(t) + \frac{f(u(x_{k+1},t)) - f(u(x_{k-1},t))}{h} = 0$$

exactly.

To turn it into a method:

1. discrete values $u_k = \bar{u}_k(t)$

2. define a numerical flux:

$$\hat{f}_{k+1/2} = f(u(x_{k+1/2}, t))$$

3. ODE solver
method "framework":
\[ \frac{du_k(t)}{dt} + \frac{\tilde{f}_{k+\frac{1}{2}}(t) - \tilde{f}_{k-\frac{1}{2}}(t)}{h_x} = 0 \]

ODE solver

FWD Euler
RK4
whatever

For \( \tilde{f}_{k+\frac{1}{2}}(t) \).
We will use \( \tilde{f}_{k+\frac{1}{2}}(t) \approx f^*(0, \cdot) \)

\[
= f^*(U_{k+1/2}^-(t), U_{k+1/2}^+(t))
\]

approx value "just to the left" of \( x_{n+1/2} \)
easiest:  \[ \tilde{u}_{k+1/2} = u_k \]
\[ u_k^{+} = u_{k+1} \]
\[
\text{let } u_{k,e} = \text{approx cell average at } t + t_e
\]

\[
\text{FWDEuler}
\]

\[
\frac{u_{k,e+1} - u_{k,e}}{h_t} + \frac{f^*(u_{k,e}, u_{k+1,e}) - f^*(u_{k-1,e}, u_{k,e})}{h_x} = 0
\]

The PDE gives us \( f(u) \)

**Definition 6.8: Consistent numerical flux function**

A numerical flux function \( f^*(u^-, u^+) \) for conservation law (6.40) is called consistent if

- \( f^*(u, u) = f(u) \) \( \forall u \)
- \( f^*(u^-, u^+) \) is Lipschitz continuous in each argument.
$$u_t + (au)_x = 0$$

$$f(u) = a \cdot u$$

Pick $f^*(u_{k-1}, u_k) = a u_k$ and $f^*(u_{k-1}, u_k) = a u_{k-1}$.

$$D = \frac{u_{k+1} - u_k}{h_t} + \frac{f^*(u_{k-1}, u_k) - f^*(u_{k-1}, u_k)}{h_x}$$

$$u_{k+1} - u_k$$

$$\frac{h_t}{h_t}$$

$$\frac{h_t}{h_t}$$

$$\text{assume } a > 0$$
if $a > 0$ : $f^*(u_{ke}, u_{k+1e}) = au_{ke}$

if $a < 0$ : $f^*(u_{ke}, u_{k+1e}) = au_{k+1e}$

Q: what if we want $f^*$ to "work" for both?

hint: use $|a|$

$$f^*(u_{ke}, u_{k+1e}) = a \left( \frac{u_{ke} + u_{k+1e}}{2} \right) - \frac{|a|}{2} (u_{k+1e} - u_{ke})$$