Recap

\[\hat{f}_{k+1/2}(t) \text{ be the numerical flux}\]
look for $f^*\left(u_{k+1/2}, u_{k+1/2}^+\right)$ for $u_t + a u_x = 0$

let $u_{k+1/2}^+, e = u_{k+1}$

$u_{k+1/2}^-, e = u_{k+1}$ for $a > 0$
for any $a \in \mathbb{R}$

$$f_n^* = a \ u_k e + a \ u_{k+1} e$$

$$= \frac{1}{2} \ a (u_{k+1} e - u_k e)$$

$$= \text{F. O. U.} \quad (\text{first-order upwind})$$

What about

$$u_t + (f(u))_x = 0 \ ?$$

$$\rightarrow \ u_t + f'(u) \ u_x = 0$$

looks like "a"
\[ u_t + a u_x = 0 \]

**Burgers**: \[ u_t + u u_x = 0 \]

**ETBS**: \[
\frac{u_{k+1} - u_k}{h_t} + a \frac{u_{k+1} - u_{k-1}}{h_x} = 0
\]

\( f'(|u|) \)
FV:

\[
\frac{u_{k+1} - u_k}{h} + f^*_x \left( \frac{f_{k+\frac{1}{2}} - f_{k-\frac{1}{2}}}{h} \right) = 0
\]

\[
f^*_{k+\frac{1}{2}} = f^*(u_k, u_{k+1})
\]

\[
= \frac{f(u_k) + f(u_{k+1})}{2} - \frac{\alpha_{k+\frac{1}{2}}}{2} (u_{k+1} - u_k)
\]

\[
\alpha_{k+\frac{1}{2}} = \max \left( \left| f'(u_k) \right|, \left| f'(u_{k+1}) \right| \right)
\]

"local" Lax-Friedrichs method or flux

LLF
So far:

- ✓ method for nonlinear f(u)
- ✓ linear in accuracy

- around §6.3

- what higher order accuracy?
- what about systems of PDE?
- what about 2D/3D?
Godunov's Method

Consider \( u_t + \left( \frac{u^2}{2} \right)_x = 0 \)

Riemann Problem

\[ u(x, 0) = \begin{cases} u^- & u \leq 0 \\ u^+ & u > 0 \end{cases} \]
Theorem 6.4: Rankine-Hugoniot relation

Let $\hat{x}(t)$ be a curve describing a jump discontinuity in a weak solution of the 1D conservation law in Equation (6.2). Then,

$$\hat{x}'(t) = \frac{f(u^+) - f(u^-)}{u^+ - u^-},$$

(6.18)

where $u^-$ and $u^+$ are the values of $u(x,t)$ to the left and to the right of the jump discontinuity.
Reconstruct solution

Average solution

Evolve the solution (as Riemann)
u(x) = m \left( x - x_{mid} \right) + u_{average}